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Abstract

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ON THE PARTIAL ORDERABILITY OF RINGS AND ALGEBRAS

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1. The question of the structural ordering of rings (algebras) without nilpotent elements, i.e., of turning them into f-rings (f-algebras) in the sense of G. Birkhoff and R. Pierce (^{1}), was considered in a previous paper by the author (^{2}). Far from always, however, is such an ordering possible. Nevertheless, it is often possible to introduce in a ring (algebra) X a sufficiently good partial order, well coordinated with the operations present in X. Here we study the question of the possibility (and also uniqueness) of introducing such partial orders in X.

2. By a **ring** X we shall everywhere mean a commutative associative ring without nonzero nilpotents, whose additive group has no torsion. By a **cone** in X we mean any additive semigroup K containing 0, closed with respect to multiplication (i.e., for K one has $K \cdot K \subset K$), isolated (i.e., such that if $nx \in K$ for some $n \in N$, where N is the set of natural numbers, then also $x \in K$), and not containing simultaneously the elements x and $-x$ for $x \neq 0$. A **partially ordered** (p.o.) **ring** is a ring X with a cone of positive elements X^+; here $x \geqslant y$ means that $x - y \in X^+$.

Definition. By a **partially ordered ring of functions** we shall mean a ring which is a subdirect product of (totally) ordered integral domains.

G. Tärper (^{12}) showed that a ring X can be turned into a p.o. ring of functions if and only if no nontrivial sums of squares are annihilated in X; in other words, if from

$$x_1^2 + \dots + x_n^2 = 0$$

it follows that

$$x_1 = \dots = x_n = 0.$$

A. Hayes (^{13}) found conditions under which a p.o. ring is isomorphic to a p.o. ring of functions. These conditions are as follows:

1) $\forall x \in X \, (x^2 \in X^+)$; 2) $\forall x \in X^+ \, \forall y \in X^+ \, \forall n \in N \, (xy - x^{2n} \in X^+)$.

He called such a ring an f^*-ring. A special case of an f^*-ring is an f-ring (commutative and containing no nonzero nilpotents; in what follows, by an f-ring we shall, without any reservation, understand only such an f-ring). We shall call the cone of positive elements of an f^*-ring (of an f-ring) an f^*-**cone** (respectively, an f-**cone**). We are interested in the question of turning a ring into a p.o. ring more "qualified" than simply an f^*-ring.

3. The annihilator of a set G in a ring X will be denoted by $X(G)$, and for $G = \{x\}$ by $X(x)$. Instead of $X(X(x))$ we shall write X_x. The binary relation $xy = 0$ in X is a disjointness relation (^{4}). The disjoint complements with respect to this relation are precisely the annihilators in X. This circumstance justifies our use of the term **component** instead of the term annihilator. We also note that if X is an f-ring, then the set of components in the sense of the theory of semiordered spaces of L. V. Kantorovich (see, for example, (^{7})) in the l-group X coincides with the set of annihilators in the ring X. A component of the form X_x will be called **principal**.

If $x = yz$ and $X(x) = X(z)$, then the element z is called a **partial** x/y.

Partially defined not for all pairs (x, y), but there may exist only one for a given pair.

Definition. A cone in a ring X is called a u-cone if it contains the squares (and hence also their sums) of all elements of X and is closed with respect to partial division.

Let us note at once that u-cones can exist only in rings in which nontrivial sums of squares are not annihilated. Obviously, every f^*-cone is a u-cone.

By the **projection** of an element x onto a component X' we mean an $x' \in X'$ (if it exists) such that

$$(x - x') \cdot X' = \{0\}.$$

Projections of an element will also be called its **fragments**. It is not hard to show that every u-cone is closed with respect to the operation of taking a fragment.

Definition. We shall call a u-cone K in a ring X a φ-cone if the following conditions A and B are fulfilled:

A. $\forall x \in X \ \forall G \subset X \ [\{xg\}(g \in G) \subset K]\&[X(G) = X(x)] \Rightarrow x \in K.$

B. $\forall x \in X \ \exists G \subset X \ [\{xg\}(g \in G) \subset K \cup (-K)]\&[X(G) = X(x)].$

A partially ordered ring X with a φ-cone X^+ will be called a φ-ring.

Theorem 1. *The set of all φ-cones in the ring X coincides with the set of all maximal u-cones and with the set of all maximal f^*-cones.*

Theorem 2. *A ring in which nontrivial sums of squares are not annihilated can be turned into a φ-ring.*

This theorem is a nontrivial strengthening of the cited result of G. Thierrin. Indeed, not every ring of functions satisfies conditions A and B. Condition A for a partially ordered ring of functions (and a φ-ring, being an f^*-ring, can be represented in this way) means that a function "glued together" from any (and not only finite) set of positive "pieces" (in particular, from a set of positive functions) is itself positive. B means that any function can be "glued together" from positive and negative "pieces."

Obviously, every f-cone is a φ-cone, but the converse, of course, is not true. In a φ-cone $K \subset X$ the condition

$$\text{B.} \quad \forall x_1 \in K \ldots \forall x_n \in K \quad (x_1 \ldots x_n = 0) \Rightarrow \inf\{x_i\} = 0.$$

holds. In an f-cone condition Γ, the converse to B, is also fulfilled.

Definition. A φ-cone satisfying in addition condition Γ will be called an f-cone. A partially ordered ring X with an f-cone X^+ will be called an f-ring.

Although every φ-ring admits a representation in the form of a partially ordered ring of functions, only for an f-ring is such a representation possible in which the lattice operations (for finite sets) are computed coordinatewise.

Definition. A ring X is called a ring with projections (with principal projections) if in X there exists the projection of any element onto any component (onto any principal component), i.e. every component (respectively every principal component) is a direct factor.

If one restricts oneself to the class of rings considered by us, then, under the additional requirement of the existence of a multiplicative identity, rings with projections turn out to be exactly Baer rings, and rings with principal projections Rickart rings (see, for example, (14)) in this class of rings.

Theorem 3. *A φ-cone in a ring with principal projections is an f-cone.*

Theorem 4. *A ring with principal projections in which nontrivial sums of squares are not annihilated can be turned into an f-ring.*

Theorem 5. A φ-cone in a ring with projections is an f-cone.

From this the following result follows immediately from (3).

Theorem 6. A ring with projections in which nontrivial sums of squares do not annihilate can be made into an f-ring.

4. In (3) it was shown that every ring can be embedded in a sufficiently good way in a ring with projections.

Definition. A subring X of a ring Y is called **pseudodense** in Y if for every $0 \neq y \in Y$, in the component Y_y there is a $0 \neq x \in X$. The ring $\overline{X}$, the smallest of all rings with projections containing the given ring X as a pseudodense subring (this means that if a ring Y contains X in a pseudodense way, then $\overline{X}$ can be embedded isomorphically in Y, leaving the elements of X fixed), is called the

completion of the ring X by means of projections, or the **P-completion** of X.

The P-completion of the ring X exists and is unique in the sense that all P-completions of X are isomorphic under an isomorphism leaving the elements of X fixed. The construction of $\overline{X}$ was given in (3). Without presenting it here, we note only that every element of $\overline{X}$ is a sum of fragments in $\overline{X}$ of elements of X. We also note that in (3) the definition of the P-completion was inaccurate: the P-completion of X was defined there simply as the smallest of the rings with projections containing X; however, in principle such a smallest ring may not exist.

Theorem 7. If X is a φ-ring and $\overline{X}$ is the P-completion of the ring X, then the partial order on X can be extended to $\overline{X}$, making the latter an f-ring. Moreover, if X is an f-ring, then one can additionally arrange that X be embedded in the f-ring $\overline{X}$ with preservation of suprema of finite sets.

Corollary. A ring in which nontrivial sums of squares do not annihilate is isomorphic to a subring of some f-ring.

5. Let us touch on the question of when a partial order with one or another property can be introduced into a ring in exactly one way.

In any ring X in which nontrivial sums of squares do not annihilate, there exist u-cones. Their intersection is also a u-cone; let K_0 be, obviously, the smallest one in X. The cone K_0 is easily described for a ring with principal projections.

Theorem 8. In order that a ring X be made into a φ-ring in exactly one way, it is necessary and sufficient that in X nontrivial sums of squares do not annihilate and that for every $x \neq 0$ there be a $0 \neq u \in K_0$ such that $0 \neq xu \in K_0 \cup (-K_0)$.

Theorem 9. In order that a ring X with principal projections be made into an f-ring (and into a φ-ring) in exactly one way, it is necessary and sufficient that in X nontrivial sums of squares do not annihilate and that every nonzero element have a nonzero fragment of the form $\pm \sum x_i^2 / \sum y_i^2$.

Theorem 10 (see (3)). In order that a ring X with projections be made into an f-ring (and into a φ-ring) in exactly one way, it is necessary and sufficient that the conditions of the preceding theorem hold for it.

6. Everything said above for the ring X carries over, without special changes, to the case of an algebra X. Here by an **algebra** is meant a ring (commutative, associative, and without nonzero nilpotents) which is an algebra over the field R of real numbers. In this case one need only require of the cone K that $x \in K$, $\lambda \in R^+$ imply $\lambda x \in R^+$.

A particular case of the problem considered in § 5 is the following. When in a given f-algebra (f-ring) X there is no second f-cone, i.e. there is no second structural order compatible in the proper way with the operations in the algebra (ring) X. In the case when there is no second structural order, we shall say that in X multiplication ⋯

defines a **structural order** (or an **f-order**). If in the given φ-algebra (φ-ring) X there is no second φ-cone, we shall say that in X multiplication defines a φ-order.

Definition. A K-algebra (K_σ-algebra) is called an f-algebra if it is simultaneously a K-space (K_σ-space) (see (7)).

From Theorem 9 there follows the following result, the first half of which was obtained in (2).

Theorem 11. *In a K_σ-algebra, multiplication defines a structural order and even a φ-order.*

Let us illustrate this theorem by an example. In functional analysis one often considers (commutative) partially ordered algebras of bounded self-adjoint operators in a Hilbert space H. A number of works ([1], [5], [6], [8–11]) have been devoted to the study of such partially ordered algebras. These are algebras of operators partially ordered in the natural way: an operator $A \geqslant 0$ means that $(Ah, h) \geqslant 0$ $(h \in H)$. This **natural partial order** usually turns the corresponding algebra of operators into an f-algebra. This is the case, for example, for a strongly closed (in the sense of operator theory) algebra, or for an algebra of the so-called type S''. By a **semiordered algebra of self-adjoint operators** we shall mean an algebra of operators that is a K-algebra with respect to the natural partial order. From Theorem 11 the following proposition follows.

Corollary. *In a semiordered algebra X of self-adjoint operators, multiplication defines a structural order (and even a φ-order). In other words, every structural order in the algebra X for which the conditions*

$$1)\ A \geqslant 0,\ B \geqslant 0 \Rightarrow AB \geqslant 0; \qquad 2)\ A^2 \wedge B^2 = 0 \Longleftrightarrow AB = 0,$$

are satisfied, coincides with the natural one.

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