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Abstract

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THEORY OF ELASTICITY

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A THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY FOR A DOMAIN BOUNDED BY TWO PARALLEL PLANES

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The problem of the equilibrium of an elastic body bounded by the planes $z = \pm h/2$ and subjected to surface forces $\mathbf{q}^+(x, y)$ and $\mathbf{q}^-(x, y)$, acting on the boundary surfaces, was reduced by V. Z. Vlasov in [1] to the solution of a system of differential equations of infinite order. In the present article we consider a more general problem in which, in addition to the surface forces $\mathbf{q}^+$ and $\mathbf{q}^-$, arbitrary body forces $\mathbf{Q}(x, y, z)$ act, and we indicate an explicit solution of this problem in the form of series whose construction is based on expansions in the eigenfunctions of a certain non-self-adjoint differential operator.

Let us first note that by the substitution

$$
\begin{aligned}
u_x &= u_0(x, y) + u_1(x, y)z + u(x, y, z), \\
u_y &= v_0(x, y) + v_1(x, y)z + v(x, y, z), \\
u_z &= w_0(x, y) + w_1(x, y)z + w_2(x, y)z^2 + w(x, y, z)
\end{aligned}
\tag{1}
$$

one can reduce the problem under consideration to the case in which, instead of the given system of forces $\mathbf{Q}$, $\mathbf{q}^+$, and $\mathbf{q}^-$, the elastic body is acted upon by a system of forces $\mathbf{P}$, $\mathbf{p}^+$, and $\mathbf{p}^-$ satisfying the conditions

$$
\int_{-h/2}^{h/2} \mathbf{P}\, dz + \mathbf{p}^+ + \mathbf{p}^- = 0, \qquad \int_{-h/2}^{h/2} \mathbf{P} z\, dz + \frac{h}{2}\left(\mathbf{p}^+ - \mathbf{p}^-\right) = 0,
$$

$$
\int_{-h/2}^{h/2} P_z \left(\frac{h^2}{4} - z^2\right) dz = 0. \tag{2}
$$

Substituting the elastic displacements u_x, u_y, and u_z, expressed by formulas (1), into the equilibrium equations of the elastic body, we obtain from conditions (2) the equations

$$
\begin{aligned}
G\left[\nabla^2 u_0 + \frac{1}{1-2\nu}\left(\frac{\partial \Delta_0}{\partial x} + 2\nu\frac{\partial w_1}{\partial x}\right)\right] + X_0 = 0,\\
G\left[\nabla^2 v_0 + \frac{1}{1-2\nu}\left(\frac{\partial \Delta_0}{\partial y} + 2\nu\frac{\partial w_1}{\partial y}\right)\right] + Y_0 = 0,\\
G\left\{\frac{h^2}{12}\nabla^2 w_1 - \frac{2}{1-2\nu}\left[\nu\Delta_0 + (1-\nu)w_1\right]\right\} = Z_1
\end{aligned}
\tag{3}
$$

for the functions u_0, v_0, w_1, and the equations

$$
\begin{aligned}
G\left(\frac{h^2}{12}\nabla^2 w_2 + \nabla^2 w_0 + \Delta_1\right) + Z_0 = 0,\\
G\left\{\frac{h^2}{12}\left[\nabla^2 u_1 + \frac{1}{1-2\nu}\left[\frac{\partial \Delta_1}{\partial x} - (1-6\nu)\frac{\partial w_2}{\partial x}\right]\right] - u_1 - \frac{\partial w_0}{\partial x}\right\} + X_1 = 0,\\
G\left\{\frac{h^2}{12}\left[\nabla^2 v_1 + \frac{1}{1-2\nu}\left[\frac{\partial \Delta_1}{\partial y} - (1-6\nu)\frac{\partial w_2}{\partial y}\right]\right] - v_1 - \frac{\partial w_0}{\partial y}\right\} + Y_1 = 0,\\
G\left\{\frac{h^2}{120}\nabla^2 w_2 + \frac{1}{6}\left[\nabla^2 w_0 + \frac{1}{1-2\nu}\left[\Delta_1 + 4(1-\nu)w_2\right]\right]\right\} + Z_2 = 0
\end{aligned}
\tag{4}
$$

for the functions w_0, u_1, v_1, w_2, where G is the shear modulus, ν is Poisson’ s ratio, and

$$
\begin{gathered}
\Delta_0 = \partial u_0/\partial x + \partial v_0/\partial y, \qquad \Delta_1 = \partial u_1/\partial x + \partial v_1/\partial y,\\
X_0 = \frac{1}{h}\left(\int_{-h/2}^{h/2} Q_x\,dz + q_x^+ + q_x^-\right), \qquad Y_0 = \frac{1}{h}\left(\int_{-h/2}^{h/2} Q_y\,dz + q_y^+ + q_y^-\right),\\
Z_0 = \frac{1}{h}\left(\int_{-h/2}^{h/2} Q_z\,dz + q_z^+ + q_z^-\right), \qquad X_1 = \frac{1}{h}\left[\int_{-h/2}^{h/2} Q_x z\,dz + \frac{h}{2}\left(q_x^+ - q_x^-\right)\right],\\
Y_1 = \frac{1}{h}\left[\int_{-h/2}^{h/2} Q_y z\,dz + \frac{h}{2}\left(q_y^+ - q_y^-\right)\right],\\
Z_1 = \frac{1}{h}\left[\int_{-h/2}^{h/2} Q_z z\,dz + \frac{h}{2}\left(q_z^+ - q_z^-\right)\right], \qquad Z_2 = \frac{10}{h^3}\int_{-h/2}^{h/2} Q_z\left(\frac{h^2}{4} - z^2\right)dz.
\end{gathered}
\tag{5}
$$

To find the elastic displacements u, v, w appearing in formulas (1), we introduce for consideration the one-dimensional non-self-adjoint boundary-value problem

$$\begin{gathered}\nu\, d^2 f/d\xi^2 - \tfrac{1}{2} d^2 g/d\xi^2 = \lambda^2(1-\nu) f,\\ 2d^2 f/d\xi^2 - (2-\nu) d^2 g/d\xi^2 = \lambda^2 (1-\nu) g,\\ -\tfrac{1}{2} < \xi < \tfrac{1}{2}, \qquad g = dg/d\xi = 0 \quad \text{for } \xi = \pm\tfrac{1}{2}.\end{gathered} \tag{6}$$

The sequence of eigenfunctions of this boundary-value problem consists of the functions $f_0^{(1)} = 1$, $g_0^{(1)} = 0$ and $f_0^{(2)} = \xi$, $g_0^{(2)} = 0$, corresponding to the eigenvalue $\lambda = 0$, and of two sequences $f_k^{(1)}(\xi), g_k^{(1)}(\xi)$ and $f_k^{(2)}(\xi), g_k^{(2)}(\xi)$, corresponding to the eigenvalues $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$, $k = 1, 2, ...$, where $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ are the nonzero roots of the equations

$$\lambda + \sin\lambda = 0, \qquad \lambda - \sin\lambda = 0, \tag{7}$$

lying in the right half-plane (the roots of equations (7) lying in the left half-plane do not give new eigenfunctions). By $f_k^{(1)*}(\xi), g_k^{(1)*}(\xi)$ and $f_k^{(2)*}(\xi), g_k^{(2)*}(\xi)$, $k = 0, 1, 2, ...$, we shall denote the eigenfunctions of the boundary-value problem

$$\begin{gathered}\nu\, d^2 f^*/d\xi^2 + 2d^2 g^*/d\xi^2 = \lambda^2(1-\nu) f^*,\\ -\tfrac{1}{2} d^2 f^*/d\xi^2 - (2-\nu) d^2 g^*/d\xi^2 = \lambda^2(1-\nu) g^*,\\ -\tfrac{1}{2} < \xi < \tfrac{1}{2}, \qquad \nu f^* + 2g^* = \nu\, df^*/d\xi + 2dg^*/d\xi = 0 \quad \text{for } \xi = \pm\tfrac{1}{2},\end{gathered} \tag{8}$$

adjoint to the boundary-value problem (6), normalized in such a way that, along with the biorthogonality conditions

$$\begin{gathered}\int_{-1/2}^{1/2} \left(f_k^{(1)} f_l^{(2)*} + g_k^{(1)} g_l^{(2)*}\right) d\xi = \int_{-1/2}^{1/2} \left(f_k^{(2)} f_l^{(1)*} + g_k^{(2)} g_l^{(1)*}\right) d\xi = 0, \quad k \ge 0,\ l \ge 0,\\ \int_{-1/2}^{1/2} \left(f_k^{(1)} f_l^{(1)*} + g_k^{(1)} g_l^{(1)*}\right) d\xi = \int_{-1/2}^{1/2} \left(f_k^{(2)} f_l^{(2)*} + g_k^{(2)} g_l^{(2)*}\right) d\xi = 0, \quad k \ne l,\end{gathered} \tag{9}$$

the conditions

$$\int_{-1/2}^{1/2} \left(f_k^{(1)} f_k^{(1)*} + g_k^{(1)} g_k^{(1)*}\right) d\xi = \int_{-1/2}^{1/2} \left(f_k^{(2)} f_k^{(2)*} + g_k^{(2)} g_k^{(2)*}\right) d\xi = 1. \tag{10}$$

A pair of functions $F(\xi)$ and $G(\xi)$, given on the interval $-\frac{1}{2} \le \xi \le \frac{1}{2}$, can be represented by the series

$$F(\xi) = c_0^{(1)} + c_0^{(2)}\xi + \sum_{k=1}^{\infty} \left[c_k^{(1)} f_k^{(1)}(\xi) + c_k^{(2)} f_k^{(2)}(\xi)\right],$$
$$G(\xi) = \sum_{k=1}^{\infty} \left[c_k^{(1)} g_k^{(1)}(\xi) + c_k^{(2)} g_k^{(2)}(\xi)\right]. \tag{11}$$

$$c_0^{(1)} = \int_{-1/2}^{1/2} \left[F(\zeta) - \frac{\nu}{2}G(\zeta)\right] d\zeta, \qquad c_0^{(2)} = 12\int_{-1/2}^{1/2} \left[F(\zeta) - \frac{\nu}{2}G(\zeta)\right] \zeta\, d\zeta,$$
$$c_k^{(j)} = \int_{-1/2}^{1/2} \left[F(\zeta) f_k^{(j)*}(\zeta) + G(\zeta) g_k^{(j)*}(\zeta)\right] d\zeta, \qquad j = 1, 2,\ k = 1, 2, \ldots. \tag{12}$$

The completeness of the biorthogonal system of eigenfunctions constructed above and the convergence of the corresponding expansions (11) follow from known results in the theory of linear differential operators (see ([2])).

Instead of the prescribed system of surface forces $\mathbf{p}^+$, $\mathbf{p}^-$ and body forces $\mathbf{P}$, we shall consider below the equivalent body forces $\mathbf{P}^*$, where

$$\mathbf{P}^* = \mathbf{P} + \delta(z, h/2)\mathbf{p}^+ + \delta(z, -h/2)\mathbf{p}^-$$
$$\left(\delta(z, z_0) = 0 \text{ for } z \neq z_0, \quad \int_{-h/2}^{h/2} \delta(z, z_0)\, dz = 1\right). \tag{13}$$

Putting successively in (11) and (12) $F = P_x^*(x, y, h\xi)$, $G = 0$; $F = P_y^*(x, y, h\xi)$, $G = 0$ and $F = 0$, $G = \int_{-1/2}^{\xi} P_z^*(x, y, h\xi')\, d\xi'$, and taking into account the equalities (2), we obtain the expansions

$$P_x^*(x, y, z) = \sum_{j=1}^{2} \sum_{k=1}^{\infty} a_k^{(j)}(x, y) f_k^{(j)}\left(\frac{z}{h}\right), \qquad P_y^*(x, y, z) = \sum_{j=1}^{2} \sum_{k=1}^{\infty} b_k^{(j)}(x, y) f_k^{(j)}\left(\frac{z}{h}\right),$$
$$P_z^*(x, y, z) = \sum_{j=1}^{2} \sum_{k=1}^{\infty} c_k^{(j)}(x, y) \varphi_k^{(j)}\left(\frac{z}{h}\right), \tag{14}$$

where

$$a_k^{(j)}(x,y) = \frac{1}{h}\left[\int_{-h/2}^{h/2} P_x(x,y,z') f_k^{(j)*}\left(\frac{z'}{h}\right) dz' + p_x^+(x,y) f_k^{(j)*}\left(\frac{1}{2}\right) + p_x^-(x,y) f_k^{(j)*}\left(-\frac{1}{2}\right)\right],$$
$$b_k^{(j)}(x,y) = \frac{1}{h}\left[\int_{-h/2}^{h/2} P_y(x,y,z') f_k^{(j)*}\left(\frac{z'}{h}\right) dz' + p_y^+(x,y) f_k^{(j)*}\left(\frac{1}{2}\right) + p_y^-(x,y) f_k^{(j)*}\left(-\frac{1}{2}\right)\right],$$
$$c_k^{(j)}(x,y) = -\frac{1}{h}\left[\int_{-h/2}^{h/2} P_z(x,y,z') \psi_k^{(j)}\left(\frac{z'}{h}\right) dz' + p_z^+(x,y) \psi_k^{(j)}\left(\frac{1}{2}\right) + p_z^-(x,y) \psi_k^{(j)}\left(-\frac{1}{2}\right)\right], \tag{15}$$

$$\varphi_k^{(j)} = \frac{dg_k^{(j)}}{d\xi}, \qquad \psi_k^{(j)} = \int_0^\xi g_k^{(j)}(\xi')\, d\xi'. \tag{16}$$

In this case the identical equality will hold

$$\sum_{j=1}^{2}\sum_{k=1}^{\infty} c_k^{(j)}(x,y) f_k^{(j)}\left(\frac{z}{h}\right) = 0. \tag{17}$$

Let us now consider the elastic displacements defined by the formulas

$$u = \frac{h}{2\pi G}\sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial x} f_k^{(j)}\left(\frac{z}{h}\right), \qquad v = \frac{h}{2\pi G}\sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial y} f_k^{(j)}\left(\frac{z}{h}\right),$$
$$w = \frac{h}{2\pi G}\sum_{j=1}^{2}\sum_{k=1}^{\infty} \Phi_k^{(j)} \frac{d}{dz}\left[g_k^{(j)}\left(\frac{z}{h}\right) - f_k^{(j)}\left(\frac{z}{h}\right)\right], \tag{18}$$

$$\Phi_k^{(j)}(x,y,x',y') = a_k^{(j)}(x',y')\, U_k^{(j)}(x-x', y-y') + \\ + b_k^{(j)}(x',y')\, V_k^{(j)}(x-x', y-y') + c_k^{(j)}(x',y')\, W_k^{(j)}(x-x', y-y') \tag{19}$$

$$U_k^{(j)}(\xi,\eta) = -\frac{1}{2h}\int_\xi^\infty W_k^{(j)}(\xi',\eta)\, d\xi' \quad \text{for } \xi > 0,$$
$$U_k^{(j)}(-\xi,\eta) = -U_k^{(j)}(\xi,\eta), \qquad V_k^{(j)}(\xi,\eta) = U_k^{(j)}(\eta,\xi),$$
$$W_k^{(j)}(\xi,\eta) = K_0\left(\frac{\lambda_k^{(j)}}{h}\sqrt{\xi^2+\eta^2}\right) \tag{20}$$

($K_0(z)$ is the Macdonald function of zero order). According to (6) and (17), in the region $|z| \le h/2$, $r \ge \varepsilon$ ($r = \sqrt{(x-x')^2 + (y-y')^2}$), the elastic displacements (18) correspond to body forces identically equal to zero, and to surface forces $\mathbf{p}$ equal to zero on the planes $z = \pm h/2$ and satisfying the limiting relations

$$\begin{aligned} \lim_{\varepsilon\to 0} 2\pi\varepsilon p_x &= \sum_{j=1}^{2}\sum_{k=1}^{\infty} a_k^{(j)}(x', y')\, f_k^{(j)}\left(\frac{z}{h}\right), \\ \lim_{\varepsilon\to 0} 2\pi\varepsilon p_y &= \sum_{j=1}^{2}\sum_{k=1}^{\infty} b_k^{(j)}(x', y')\, f_k^{(j)}\left(\frac{z}{h}\right), \\ \lim_{\varepsilon\to 0} 2\pi\varepsilon p_z &= \sum_{j=1}^{2}\sum_{k=1}^{\infty} c_k^{(j)}(x', y')\, \varphi_k^{(j)}\left(\frac{z}{h}\right) \end{aligned} \tag{21}$$

on the cylindrical boundary surface $r = \varepsilon$. In accordance with the expansions (14), formulas (18) determine the elastic displacements caused by body forces $\mathbf{P}^*(x', y', z)\delta(x - x', y - y')$ $(\delta(\xi, \eta) = 0$ for $\xi^2 + \eta^2 \neq 0$, $\iint \delta(\xi, \eta)\, d\xi\, d\eta = 1)$.

Thus, the elastic displacements caused by body forces $\mathbf{P}^*(x, y, z)$ acting in the region $x, y \in S$, $|z| \leq h/2$, can be found from the formulas

$$\begin{aligned} u(x, y, z) &= \frac{h}{2\pi G} \iint_S \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial x}\, f_k^{(j)}\left(\frac{z}{h}\right) dx'\, dy', \\ v(x, y, z) &= \frac{h}{2\pi G} \iint_S \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{\partial \Phi_k^{(j)}}{\partial y}\, f_k^{(j)}\left(\frac{z}{h}\right) dx'\, dy', \\ w(x, y, z) &= \frac{h}{2\pi G} \iint_S \sum_{j=1}^{2}\sum_{k=1}^{\infty} \Phi_k^{(j)} \frac{d}{dz}\left[g_k^{(j)}\left(\frac{z}{h}\right) - f_k^{(j)}\left(\frac{z}{h}\right)\right] dx'\, dy'. \end{aligned} \tag{22}$$

As the distance from the region in which the body forces $\mathbf{P}^*$ act increases, the elastic displacements u, v, w rapidly decrease in accordance with the exponential law of decrease of the Macdonald function $K_0(z)$ as $\operatorname{Re} z \to +\infty$.

The solution indicated by us of the three-dimensional problem of the theory of elasticity for a region bounded by the planes $z = \pm h/2$ opens the possibility of a correct calculation of local deformations in elastic plates. The method applied above of expansion in eigenfunctions of the boundary-value problem (6) can also be used for a refined calculation of edge effects arising in an elastic plate of a prescribed configuration.

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2. E. Kamke, *Handbook of Ordinary Differential Equations*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.