ON THE NONLINEAR INTERACTION OF RESONATING OSCILLATORS

PHYSICS

1970

SovietRxiv

Abstract

Full Text

UDC 531.561.2

PHYSICS

L. G. KHAZIN, F. Kh. TSELMAN

ON THE NONLINEAR INTERACTION OF RESONATING OSCILLATORS

(Presented by Academician A. Yu. Ishlinskii on 30 XII 1969)

Recently the problem of the nonlinear interaction of oscillators has been attracting increasing attention (see ([6–10])). If a system has no resonances, then by means of a special change of variables it can be reduced, to any finite order, to a system of independent "quasi-oscillators" (see ([2–4])). Thus, in essence, the "interacting" systems are resonant systems. Apparently, the problem of the nonlinear interaction of resonating oscillators was first considered in the work of G. Gorelik and A. Witt ([1]). In it the oscillations of a system with two degrees of freedom (a plane elastic pendulum) were studied in the case of a frequency ratio 2 : 1.

The present work is devoted to the study of resonant Hamiltonian systems with n degrees of freedom.

1. Statement of the problem. Consider a conservative system describing n coupled oscillators, with Hamiltonian

$$H(p,q) = H^{(2)}(p,q) + H^{(3)}(p,q) + \dots + H^{(\alpha)}(p,q) + \dots \quad (1)$$

Here $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$, and $H^{(\alpha)}(p,q)$ is a homogeneous polynomial in the variables p, q of degree α. In the linear approximation the system is a set of n independent oscillators*, i.e.

$$H^{(2)}(p,q) = \frac{1}{2}\sum_{\nu=1}^{n} \beta_\nu \left(p_\nu^2 + q_\nu^2\right) \quad (\beta_\nu > 0), \quad (2)$$

where $\pm i\beta_\nu$ are the eigenvalues of the linearized system with Hamiltonian (1). We shall assume that among the eigenvalues there are no multiple ones, i.e. $\beta_\alpha \neq \beta_\gamma$ if $\alpha \neq \gamma$.

Definition. The system (1) is said to **possess a resonance** if

$$\sum_{\alpha=1}^{n} k_\alpha \beta_\alpha = 0, \tag{3}$$

where k_α are integers. The number $k = \sum_{\alpha=1}^{n} |k_\alpha|$ is called the **order of the resonance****. The vector $k = (k_1, \dots, k_n)$ is called the **resonance vector**.

In the present work systems with a single resonance relation (3) will be considered. The behavior of the system will be investigated to an order equal to the order of the resonance.

* Any Hamiltonian system can be brought to this form in the case of positive definiteness of $H^{(2)}(p,q)$.

** The absence of multiple frequencies, in particular, means that $k \geq 3$.

p. 2. **Model systems.** Let the order of the lowest resonance, determined by relation (3), be equal to $m > 2$. It is known (see, for example, ([3])) that then there exists a real polynomial canonical change of variables of degree $m-1$, $(p, q \to \xi, \eta)$, such that the system with Hamiltonian (1) is transformed into a system with Hamiltonian

$$2H = \sum_{\nu=1}^{n} \beta_\nu \rho_\nu + H_2(\rho) + \dots + H_\mu(\rho) + \Gamma_m(\rho, \psi) + R(\rho, \varphi). \tag{4}$$

Here $\rho_\alpha, \varphi_\alpha$ $(\alpha = 1, \dots, n)$ are polar coordinates; $\xi_\alpha = \sqrt{\rho_\alpha}\sin\varphi_\alpha$; $\eta_\alpha = \sqrt{\rho_\alpha}\cos\varphi_\alpha$; $\rho = (\rho_1, \dots, \rho_n)$; $\varphi = (\varphi_1, \dots, \varphi_n)$; $\psi = \sum_{\alpha=1}^{n} k_\alpha \varphi_\alpha$ is the resonant phase; $H_\nu(\rho)$ is a homogeneous polynomial of degree ν in the variables ρ; $\mu = [m/2] - 1$; $k = \sum_{\alpha=1}^{n} |k_\alpha| = m$;

$$\Gamma_m(\rho, \psi) = \begin{cases} 2A\sqrt{\rho^{|k|}}\cos\psi, & \text{if } k = 2d+1,\ d \geqslant 1, \\ 2A\sqrt{\rho^{|k|}}\cos\psi + A_l \rho^{|l|}, & \text{if } k = 2d,\ d \geqslant 2, \end{cases}$$

where $\rho^{|k|} = \rho_1^{|k_1|}\rho_2^{|k_2|}\cdots\rho_n^{|k_n|}$, $l = (l_1, \dots, l_n)$ is an integer vector;

$$l = \sum_{\alpha=1}^{n} |l_\alpha| = d; \qquad A_l \rho^{|l|} = \sum_{|l|=d} A_{l_1 l_2 \dots l_n} \rho_1^{|l_1|}\rho_2^{|l_2|}\cdots\rho_n^{|l_n|}.$$

$R(\rho, \varphi)$ has degree in ρ greater than m. The Hamiltonian

$$\Gamma = 2H - R(\rho, \varphi) \tag{5}$$

coincides, up to terms of order higher than m in the variables ρ, with the "normal form" (see ([5])) of the Hamiltonian $2H$.

The system described by this Hamiltonian will be called an m-model system.

It is not difficult to verify that

$$J_\alpha = \rho_\alpha - \frac{k_\alpha}{k_1}\rho_1 \qquad (\alpha = 2, \dots, n), \tag{6}$$

$$F = \Gamma - \sum_{\alpha=1}^{n} \beta_\alpha \rho_\alpha \tag{7}$$

are independent integrals of system (5).

The Hamiltonian (5) depends essentially on $n+1$ variables ρ_i $(i = 1, \dots, n)$ and the resonant phase ψ. With the aid of the integrals (6), (7) one can eliminate n variables and obtain an autonomous first-order differential equation for one of the variables ρ_i*.

Remark 1. Since in the first approximation ρ_i corresponds to the energy of the i-th oscillator, in many problems, in order to describe the interaction of the oscillators, it is sufficient to study a single equation for ρ_i. The remaining ρ_i are easily obtained from (6). After determining the ρ_k, the phases φ_l are found by quadratures.

p. 4. **Third-order resonances.** They correspond to one of the following relations between the frequencies: 1) $\beta_1 = 2\beta_2$, 2) $\beta_1 + \beta_2 = \beta_3$.

The resonance $\beta_1 = 2\beta_2$. For $n = 2$ this is the case considered in work ([1]). The 3-model system is determined by the Hamiltonian

$$\Gamma = \beta_2(2\rho_1 + \rho_2) + 2A\sqrt{\rho_1\rho_2^2}\cos\psi \qquad (\psi = \varphi_1 - 2\varphi_2). \tag{8}$$

* We note that system (5), in the presence of the integrals (6), (7), is integrable (Liouville theorem).

For ρ_2, with the aid of integrals of the form (6), (7), for system (8) one obtains the equation

$$\dot\rho_2 = \pm 2\sqrt{2A^2\rho_2^2(J - \rho_2) - F^2} \quad (J = 2\rho_1 + \rho_2). \tag{9}$$

A qualitative investigation of such an equation in several other variables was carried out in ([1]). Study of equations (9) in the phase plane $\rho, \dot\rho$ (depending on the value of F) shows that: 1) generally speaking, a periodic "pumping of energy" * between the oscillators takes place; 2) there exists a stationary regime in which there is no energy pumping (center); 3) there exists a regime in which one of the oscillators asymptotically comes to rest ($\rho_2 \to 0$) (separatrix).

Remark 2. The same picture of energy pumping between resonating oscillators will also occur in the case when they enter into a system of n coupled oscillators, if there are no other resonance relations.

Resonance $\beta_1 + \beta_2 = \beta_3$. In this case the integrals of the 3-model system

$$\Gamma = \sum_{\nu=1}^{n} \beta_\nu \rho_\nu + 2A\sqrt{\rho_1\rho_2\rho_3}\cos\psi, \qquad \psi = \varphi_1 + \varphi_2 - \psi_3$$

of the form (6), (7) make it possible to write down the equation for ρ_1

$$\dot{\rho}_1 = \pm\sqrt{4A^2\rho_1(J_2+\rho_1)(J_3-\rho_1) - F^2}. \tag{10}$$

Investigation of equation (10) shows that in the case under consideration there is a special regime when two oscillators asymptotically come to rest.

§ 5. Resonances of the fourth order. Let us dwell on a fourth-order resonance defined by the relation $\beta_1 = 3\beta_2$. In the case of 2 degrees of freedom (see Remark 2), the 4-model system with Hamiltonian

$$\Gamma = \beta_2(3\rho_1 + \rho_2) + 2A\sqrt{\rho_1\rho_2^3}\cos\psi + A_l\rho^{|l|} \qquad (l = 2)$$

has the integrals $J = 3\rho_1 + \rho_2$, $F = \Gamma - \beta_2 J$. Investigation of the equation for ρ_1 shows that in this case complete pumping of energy (at least one of the $\rho_i \to 0$) is possible only under very stringent restrictions on the coefficients of the system A_l.

Remark 3. Higher-order resonances in Hamiltonian systems with one resonance relation are investigated in an analogous way.

Remark 4. The difference in the magnitude of the frequencies participating in the resonance affects the character of the singular points of the equation for ρ_i.

Institute of Applied Mathematics
Academy of Sciences of the USSR
Moscow

Institute of Control Problems
(Automation and Telemechanics)
Moscow

Received
24 XII 1969

CITED LITERATURE

[1] G. Gorelik, A. Witt, ZhTF, 3, issue 2–3, 294 (1933).
[2] J. D. Birckhoff, *Dynamical Systems*, M.—L., 1941.
[3] J. Moser, Mem. Am. Math. Soc., 81 (1968).
[4] K. L. Zigel, *Lectures on Celestial Mechanics*, M., 1959.
[5] A. P. Bryuno, DAN, 174, No. 5, 1003 (1967).
[6] F. L. Chernousko, Zhurn. Vychislit. Matem. i Matem. Fiz., 3, No. 1, 131 (1963).
[7] A. P. Markeev, Kosmicheskie Issledovaniya, 5, issue 3, 365 (1967).
[8] A. P. Torzhevskii, Kosmicheskie Issledovaniya, 6, issue 1, 58 (1968).
[9] R. F. Ganiev, V. O. Kononenko, Mekh. Tverd. Tela, No. 3, 3 (1968).
[10] R. Pringle jr., AIAA J., 6, No. 7, 1217 (1968).

[*] Periodic change of ρ_i (see Remark 1).

Note: Figure translations are in progress. See original paper for figures.