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Abstract

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ON SOME APPLICATIONS OF SPECIAL COORDINATE CONDITIONS IN A. EINSTEIN' S THEORY OF GRAVITATION

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As is known, A. Einstein' s gravitational equations

$$G_{ik} = R_{ik} - {}^{1}/{}_{2} g_{ik} R = -\gamma T_{ik} \tag{1}$$

possess the remarkable property that, by virtue of the well-known differential identities

$$\delta G^{ik} / \delta x^{k} = 0 \tag{2}$$

they do not form a complete system, determining from the 10 quantities g_{ik} only 6. The importance of the identities (2) is due above all to the fact that, by taking the contracted covariant derivative of both sides of (1), they lead to the conservation laws for energy-momentum, $\delta T^{ik}/\delta x^{k} = 0$. In addition, these identities, by allowing freedom in the choice of 4 quantities g_{ik} out of 10, ensure the choice of arbitrary coordinate systems, which is usually done by adding to the system of equations (1) four noncovariant differential equations for g_{ik}. These latter are usually called coordinate conditions. Their choice is dictated by the convenience of considering and solving particular problems. Thus, for example, in astronomy the coordinate conditions of V. A. Fock (^{1}), which define the so-called "harmonic" coordinates, have proved very useful.

For further purposes let us take coordinate conditions of the special form

$$t_{k}^{4} = 0 \ (t_{4}^{k} = 0) \quad \text{or} \quad t^{4k} = 0, \tag{3}$$

where t_{i}^{k} or t^{ik} are pseudotensors of the energy-momentum of the gravitational field, introduced by variational methods, as was done, for example, by A. Einstein (^{2}), who introduced the Hamilton function and obtained his expression for t_{i}^{k}, or by directly using the conservation laws following from (1), as was done by L. Landau and E. Lifshitz (3), who gave their pseudotensor t_L^{ik}. It is very essential that t_i^k or t^{ik} are not tensors; consequently, equations (3) are not covariant, and that these equations, owing to the choice of expressions for t_i^k or t^{ik} in agreement with equations (1), do not contradict the latter.

At first glance, the coordinate conditions (3) seem very extravagant and difficult to use because of their nonlinearity. However, it turns out that they are automatically satisfied for a number of exact solutions, such as: Schwarzschild' s solution for a point mass m, including also the point $r = 0$ ($t_i^k = 0$), Einstein and Rosen' s (4) for gravitational waves in vacuum ($t_i^k = 0$), Peres' (5) ($t_i^k = t_L^{ik} = 0$).

Apparently, in a number of cases the transition to coordinates satisfying conditions (3) can be carried out quite simply by an appropriate coordinate transformation. Thus, for example, solutions of the equations for plane waves propagating in the direction x^1 in the weak-field approximation are given by the relations

$$g_{ik} = \delta_{ik} + h_{ik}, \qquad \delta_{ik} = \begin{cases} 0, & i \neq k, \\ 1, & i = k, \end{cases} \tag{4}$$

$$|h_{ik}| \ll 1, \qquad h_{ik} = a_{ik} f(x^1 + ix^4) = a_{ik} f(\xi), \tag{4a}$$

where only the following are nonzero:

$$h_{22} = h_{33} \neq 0, \qquad h_{23} \neq 0. \tag{5}$$

As Eddington showed, only such waves have a nonzero Riemann curvature tensor $R_{iklm} \neq 0$.

According to Einstein (7), lowering all indices, owing to the condition $|h_{ik}| \ll 1$, for such waves

$$4\chi t_{ik} = \frac{\partial h_{lm}}{\partial x^i} \frac{\partial h_{lm}}{\partial x^k} - \frac{\delta_{ik}}{2} \left(\frac{\partial h_{lm}}{\partial x^n} \right)^2, \tag{6}$$

and only the following components of the gravitational pseudotensor will be nonzero:

$$t_{11} = \frac{t_{14}}{i} = -t_{44} = \frac{a_{lm}^2}{4\chi} \left(\frac{\partial f}{\partial \xi} \right)^2. \tag{7}$$

The transition to the coordinates defined by equations (3) is easily carried out by the linear coordinate transformation

$$x_1' = \alpha x_1 + i\beta x_4, \qquad x_4' = \gamma(x_4 - i x_1), \tag{8}$$

with determinant $\Delta = \gamma(\alpha - \beta) \neq 0$, where α, β, and γ are real numbers.

Obviously, in this coordinate system, by virtue of (3), plane gravitational waves have no energy and do not carry momentum or energy.

Proceeding from the possibility of realizing solutions of the gravitational equations (1) in coordinate systems defined by equations (3), let us try to draw some physical conclusions:

A. For a gravitational field in vacuum ($R_{iklm} \neq 0$), defined by a solution of equation (1) without matter in any coordinate systems that pass into Galilean ones at infinity, the 4-vector of momentum-energy of the gravitational field

$$P_i = \int t_i^4 \, dv = 0, \qquad P^i = -\int g t^{i4} \, dv = 0, \tag{9}$$

where $dv = dx\,dy\,dz$ is the spatial element of volume. This assertion is true for the choice of coordinates (3). But according to Einstein (^{8}), in accordance with (^{3}), for systems with Galilean conditions at infinity,

$$P^i = \int -g\,(T^{i4} + t^{i4})\,dv = \text{const},$$

$$P_i = \int (\sqrt{-g}\, t_i^4 + T_i^4)\,dv = \text{const} \tag{10}$$

is a generally covariant 4-vector, as will also be P_i defined by (9), if in (10) one puts $T^{ik} = 0$. Thus, if equations (9) are satisfied in one coordinate system with Galilean conditions at infinity, then they will hold in all such coordinate systems. Consequently, equations (9) have real physical meaning. Moreover, assertion A is thereby proved.

To illustrate it one may use Peres' exact solution:

$$dS^2 = dx^2 + dy^2 + dz^2 - dt^2 + 2f(dz + dt)^2,$$

$$f = \psi(z + t)\varphi(x, y), \tag{11}$$

$$f_{xx} + f_{yy} = \gamma T_{33} = \gamma T_{34} = \gamma T_{44}.$$

As has already been noted, for this solution ${}_E t_i^k$ and ${}_L t^{ik} = 0$. Therefore equations (9) are satisfied for solution (11), if in them one sets $T_{33} = T_{34} =$

Fig. 1

Figure 1: Fig. 1

$= T_{44} = 0$ and take for f a solution of the Laplace equation

$$f_{xx} + f_{yy} = 0$$

with Galilean conditions at infinity. Such a solution may be taken, for example, in the form

$$f = \frac{xy}{(x^2+y^2)^2} e^{-(z+t)^2}. \tag{12}$$

Arnowitt, Deser, and Misner $^{(9)}$ point out that if for vacuum solutions $(T^{ik} = 0)$ of equations (1) the boundary conditions are taken in the form (4) with $h_{ik} \sim 1/r$ at infinity, then the integrals (9) will not be equal to zero. But it is well known $^{(3)}$ that in this case, according to (1), there must exist a material source with $T^{ik} \neq 0$ in a finite region of space-time, or in the form of a δ-function for T^{ik} (the Schwarzschild solution). Therefore vacuum solutions of the gravitational equations (1) with $h_{ik} \sim 1/r$ at infinity do not exist, quite apart from the fact that such solutions contradict equation (9) for vacuum solutions.

Fig. 1

Another example of the fulfillment of equations (9) may be a solution of (1) in the weak-field approximation of type (4) in the form of a wave packet with Galilean conditions at infinity $^{(10)}$, in which

$$h_{\mu\nu} = f_{\mu\nu}(r + ix_4)/r, \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{13}$$

The importance of statement A from the physical point of view is seen from the fact that the Hamiltonian function $H = t_4^4$, and according to (9) for the gravitational field in vacuum $H = 0$.

B. A material system with $T_{ik} \neq 0$, described by solutions (1) with Galilean conditions at infinity, cannot pass into a state with the same boundary conditions by decreasing or increasing its energy through gravitational radiation.

It is convenient to give a proof of this statement on the basis of the discussion of one example considered in $^{(11)}$.

Let some material system with Galilean conditions at infinity and with four-vector

$$P_i^{(a)} = \int_a \left(\sqrt{-g}\, T_i^4 + t_i^4\right) dv$$

at the moment $t = 0$, evolving in time, emit a packet of gravitational waves also with Galilean conditions at infinity and with four-vector

$$P_i^{(g)} = \int_g \sqrt{-g}\, t_i^4 \, dv.$$

After emitting the packet it will again, at the moment $t = T \gg 0$, be a system with the same conditions at infinity and with four-vector

$$P_i^{(b)} \int_b (\sqrt{-g}\, T_i^4 + t_i^4) \; dv$$

(Fig. 1). Then, by virtue of the conservation laws, when the coordinate system is chosen in the region (g) satisfying condition (3),

$$P_i^{(g)} = P_i^{(a)} - P_i^{(b)} = \sqrt{-g} \int_g t_i^4 \, dv = 0. \tag{14}$$

Since relation (14) is generally covariant, it will hold in any coordinate systems for regions of space-ti-

of the metrics (a), (b), and (g) with the Galilean conditions at infinity. Consequently, the equation has objective physical content, independent of the choice of coordinate systems. Statement (B) is thereby proved.

It is interesting that the authors of the example used here tried to show precisely the opposite, making an incorrect estimate of $P_i^{(g)} = \sqrt{-g} \int t_i^4 \, dv$ by means of the formula for quadrupole radiation in the weak-field approximation (^{3}).

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Note: Figure translations are in progress. See original paper for figures.