ON THE SMOOTHNESS OF A FUNCTION OF THREE VARIABLES OF BOUNDED VARIATION

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ON THE SMOOTHNESS OF A FUNCTION OF THREE VARIABLES OF BOUNDED VARIATION

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In this paper, $t = F(\eta)$ denotes a function of 3 variables, continuous in R_3 and equal to zero outside the closed ball Q. E_{t_0} is the level set $t = t_0$ of the function $F(\eta)$. If E is a set in R_k $(k \leq 3)$, then its k-dimensional Lebesgue measure will be denoted by $\mu_k(E)$. Let $E \subset R_3$ be a point set; by $\operatorname{mes}_2 E$ we shall denote its area, i.e., its two-dimensional Hausdorff measure. By P_{x_0} (respectively P_{y_0}, P_{z_0}) we shall denote the plane $x = x_0$ (respectively $y = y_0, z = z_0$).

Definitions. Let $U \subset Q$ be a closed ball and let Σ be an arbitrary finite system of pairwise disjoint closed balls $U_1, \dots, U_n$ contained in Q.

1. Let $h_{0,U}(t)$ be the number of components of the level set E_t wholly belonging to the ball U (it is allowed that $h_{0,U}(t) = +\infty$). The **partial lower variation** of $F(\eta)$ in the ball U is

$$V_{0,U}(F) = \int_{-\infty}^{+\infty} h_{0,U}(t)\, dt.$$

The **partial lower variation** of $F(\eta)$ with respect to the system Σ is

$$V_{0,\Sigma}(F) = \sum_{i=1}^{n} V_{0,U_i}(F).$$

If $U \equiv Q$, then $W_0(F) \equiv V_{0,Q}(F)$ is called the **total lower variation** of $F(\eta)$.

2. Let $v_U(t, x_0)$ be the number of components of the set $E_t \cap P_{x_0}$ wholly belonging to the ball U. The **partial x-length of the level set E_t** in the ball U is

$$h_{1,x,U}(t) = \int_{-\infty}^{+\infty} v_U(t, x)\, dx.$$

Analogously, $h_{1,y,U}(t)$ and $h_{1,z,U}(t)$ are defined—the partial y- and z-lengths of E_t in the ball U. Put

$$h_{1,U}(t) = h_{1,x,U}(t) + h_{1,y,U}(t) + h_{1,z,U}(t).$$

3. The **partial middle variation** of $F(\eta)$ in the ball U is

$$V_{1,U}(F) = \int_{-\infty}^{+\infty} h_{1,U}(t)\,dt,$$

and the **partial middle variation** of $F(\eta)$ with respect to the system Σ is the sum

$$V_{1,\Sigma}(F) = \sum_{i=1}^{n} V_{1,U_i}(F).$$

If $U \equiv Q$, then $W_1(F) \equiv V_{1,Q}(F)$ will be called the **total middle variation** of the function $F(\eta)$.

4. Put

$$h_{2,U}(t) = \mathrm{mes}_2(E_t \cap U).$$

The **upper partial variation** of $F(\eta)$ in U is

$$V_{2,U}(F) = \int_{-\infty}^{+\infty} h_{2,U}(t)\,dt.$$

The **upper partial variation** with respect to the system Σ is

$$V_{2,\Sigma}(F) = \sum_{i=1}^{n} V_{2,U_i}(F).$$

For $U \equiv Q$, we shall call $W_2(F) \equiv V_{2,Q}(F)$ the **total upper variation** of the function $F(\eta)$.

All functions occurring in definitions 1–4 under the integral sign are Lebesgue measurable. The integral is everywhere understood as a Lebesgue integral.

Theorem 1. For any system Σ of pairwise nonintersecting closed balls $U_i \subset Q$ $(i = 1, 2, \dots, n)$, one has

$$V_{s,\Sigma}(F) \le W_s(F) \qquad (s = 0, 1, 2).$$

Theorem 2. Let $U \subset Q$ and $V \subset Q$ be concentric closed balls of radii δ and 8δ, respectively, and suppose that for $t = t_0$, $E_{t_0} \cap U \neq \emptyset$. Then at least one of the three inequalities holds:

$$h_{0,V}(t) \geq 1, \qquad h_{1,V}(t_0) \geq \delta/2, \qquad h_{2,V}(t_0) \geq \delta^2/4.$$

In the proof of Theorem 2 the following lemmas are used:

Lemma 1. Under the assumptions of Theorem 2, let $L \subset V$ denote a continuum intersecting both U and the sphere S_V of the ball V. Let K_χ denote the surface of the cube whose faces are parallel to the coordinate planes, whose center coincides with the center of the balls U and V, and whose edge is equal to 2χ.

By a **marked plane** π_χ we shall mean any plane that contains a face of K_χ intersecting L. Then the marked planes π_χ, for $\chi \in [\delta, 4\delta]$, cut out on the coordinate axes closed sets F_x, F_y, and F_z, respectively. For at least one of these sets the linear Lebesgue measure is not less than δ.

Lemma 2. Let F_x be the set defined under the assumptions of Lemma 1, with $\mu_1(F_x) \geq \delta$. Denote by E the set of all such $x_0 \in F_x$ for each of which there exists at least one component of the set $L \cap P_{x_0}$ that does not intersect S_V. Then at least one of the inequalities is true:

$$\mu_1(E) \geq \delta/2, \qquad \text{mes}_2\, L \geq \delta^2/4.$$

A result well known from topology is used essentially in the work:

Separation theorem. If a closed set F separates the points a and β in the ball U, then there exists at least one component of the set F separating these points in U.

Theorem 3 (on the differentiability of a function of bounded variation). Let all three variations $W_0(F)$, $W_1(F)$, and $W_2(F)$ of the function $F(\eta)$ be bounded. Then $F(\eta)$ has a complete differential almost everywhere.

Proof. Let E be the set of points of nondifferentiability of the function $F(\eta)$. Suppose that $\mu_3(E) = \tau > 0$ (the set E is measurable B). Let E^* be the set of all points $\alpha \in E$ for which

$$\sup_{\eta \in Q} \left| \frac{F(\eta) - F(\alpha)}{\rho(\eta, \alpha)} \right| = +\infty.$$

By a theorem of V. V. Stepanov, $\mu_3(E^*) = \tau$. Fix $k > 0$. From the continuity of $F(\eta)$ it follows that in any neighborhood of a point $\alpha \in E^*$ there exists a point β such that

$$|F(\beta) - F(\alpha)| > k\rho(\beta, \alpha). \qquad (*)$$

For each point $\alpha \in E^*$ construct a sequence of points $\{\eta_{n,\alpha}\}$ converging to α, satisfying relation $(*)$, with $\rho(\alpha, \eta_{n,\alpha}) < 1$.

The system of balls $\{V_{n,\alpha}\}$ with centers at $\alpha \in E^*$ and radii $8\rho(\alpha, \eta_{n,\alpha})$ forms a Vitali covering of the set E^*. From it select a system Σ of pairwise nonintersecting closed balls $\{U_i\}$ $(i = 1, 2, \dots, l)$ contained in Q, with centers at the points α_i and radii $8\delta_i$, where $\delta_i = \rho(\alpha_i, \beta_i)$, and the points α_i, β_i satisfy relation $(*)$, such that

$$\mu_3\left(E^* \cap \sum_{i=1}^{l} U_i\right) > \frac{1}{3}\tau.$$

Then we have:

$$\frac{4}{3}\pi \sum_{i=1}^{l} (8\delta_i)^3 > \frac{1}{3}\tau, \qquad \sum_{i=1}^{l} (\delta_i)^3 > \frac{1}{8000}\tau, \qquad |F(\beta_i) - F(\alpha_i)| > k\delta_i > k\delta_i^2.$$

Let T_i be the set of all t lying between $F(\alpha_i)$ and $F(\beta_i)$. For each $t \in T_i$, the set $E_t \cap U_i$ separates the points α_i and β_i in the ball U_i. Consequently, in the set $E_t \cap U_i$ there exists at least one component K_i separating α_i and β_i in the ball U_i. Let S_U be the sphere of the ball U_i. If $K_i \cap S_{U_i} = \emptyset$, then assign K_i to the 1st class. If $K_i \cap S_{U_i} \neq \emptyset$ and $\operatorname{mes}_2 K_i <$

$< \frac{1}{4}\delta_i^2$, then we assign K_i to the 2nd class. If $K_i \cap S_{U_i} \neq \emptyset$ and $\operatorname{mes}_2 K_i \geq \frac{1}{4}\delta_i^2$, then we assign K_i to the 3rd class.

Let $T_{1,i}$ be the set of all $t \in T_i$ for which at least one component of the set $E_t \cap U_i$ separating α_i and β_i in the ball U_i belongs to the 1st class.

Denote by $T_{2,i}$ the set of all $t \in T_i$ for which in the set $U_i \cap E_t$ there exists at least one component of the 2nd class separating α_i and β_i. Put $T_{3,i} = T_i \setminus (T_{1,i} \cup T_{2,i})$. The sets $T_{1,i}, T_{2,i}$, and $T_{3,i}$ are B-measurable, and the measure of at least one of them is not less than $\frac{1}{3}k\delta_i$. Then, by Theorem 2, we have:

$$V_{0,\Sigma}(F) \geq \sum_{i=1}^{l} \mu_1(T_{1,i}) > \sum_{i=1}^{l} \mu_1(T_{1,i})\delta_i^2, \qquad V_{1,\Sigma}(F) \geq \frac{1}{2}\sum_{i=1}^{l} \mu_1(T_{2,i})\delta_i >$$

$$> \frac{1}{2}\sum_{i=1}^{l} \mu_1(T_{2,i})\delta_i^2, \qquad V_{2,\Sigma}(F) \geq \frac{1}{4}\sum_{i=1}^{l} \mu_1(T_{3,i})\delta_i^2.$$

Hence, taking Theorem 1 into account, we have

$$W_0(F) + W_1(F) + W_2(F) \geq V_{0,\Sigma}(F) + V_{1,\Sigma}(F) + V_{2,\Sigma}(F) \geq$$

$$\geq \frac{1}{12} k \sum_{i=1}^{l} \delta_i^3 > \frac{k\tau}{10^5}.$$

Since $k > 0$ was chosen arbitrarily, we conclude that at least one of the three variations $F(\eta)$ is unbounded. This contradicts the conditions of the theorem. We shall show that boundedness of any two variations of a continuous function $F(\eta)$ is not a sufficient condition for differentiability almost everywhere of this function.

In the examples considered below, we shall regard $F(\eta)$ as defined in the closed ball R of unit radius with center at the origin O.

Example 1. Inscribe in R a cube Q, into which we place an open ball U_0 of radius 2^{-2}. Divide Q into a system of equal cubes of the 1st rank so that the sum of the volumes of the cubes of the 1st rank intersecting U_0 does not exceed $\frac{8}{3}\pi \cdot 2^{-6}$.

Let $Q_{1,1}, \dots, Q_{1,k_1}$ be the cubes of the 1st rank not intersecting U_0. In each cube $Q_{1,i}$ place an equal open ball $U_{1,i}$ of radius not exceeding k_1^{-1}, and such that the sum of the volumes of the balls $U_{1,i}$ does not exceed $\frac{4}{3}\pi \cdot 2^{-9}$. Let

$$\sigma_1 = \bigcup_{i=1}^{k_1} U_{1,i} \quad \text{and} \quad P_1 = \bigcup_{i=1}^{k_1} Q_{1,i}.$$

Divide P_1 into a system of equal cubes of the 2nd rank so that the sum of the volumes of the cubes of the 2nd rank intersecting σ_1 does not exceed $\frac{8}{3}\pi \cdot 2^{-9}$. Let $Q_{2,1}, \dots, Q_{2,k_2}$ be the cubes of the 2nd rank not intersecting σ_1. In each cube $Q_{2,i}$ place an equal open ball $U_{2,i}$ of radius not exceeding k_2^{-1}, and such that the sum of the volumes of the balls $U_{2,i}$ does not exceed $\frac{4}{3}\pi \cdot 2^{-12}$. Let

$$\sigma_2 = \bigcup_{i=1}^{k_2} U_{2,i} \quad \text{and} \quad P_2 = \bigcup_{i=1}^{k_1} Q_{2,i}.$$

Continue this process without bound. Define $F(\eta)$ as follows: 1) $F(\eta) = 0$ for $\eta \in R \setminus \bigcup_{n=1}^{\infty} \sigma_n$;

2) $F(\eta) = 2^{-n}$, if η is the center of a ball of the system σ_n $(n = 1, 2, \dots)$; 3) $F(\eta)$ is linear on the radii of each of the balls of these systems. It is not difficult to see that $F(\eta)$ is continuous, while the variations $W_1(F)$ and $W_2(F)$ are bounded. Let

$$\Omega = \bigcap_{n=1}^{\infty} P_n.$$

Then $\mu_3(\Omega) > 0$. At the points of the set Ω the function $F(\eta)$ is not differentiable.

Example 2. Let Q_1 and Q_2 be parts of right circular cylinders of radii $\frac{1}{2}$ and $\frac{1}{4}$, respectively, with generators parallel to the z-axis, bounded by the planes $z = -\frac{1}{2}$ and $z = \frac{1}{2}$, and with the centers of symmetry of Q_1

and Q_2 coincide with O. Denote by A_0 the interior of the disk A—the intersection of Q_2 with the plane $z = 0$.

Draw in the disk A_0 a simple arc L_1 so that in the neighborhood of any point $a \in A_0$ of radius 10^{-2} there exists at least one point of L_1. Let $P_1 \supset L_1$ be such an open strip that $\mu_2(P_1) < 10^{-2}$ and $\overline{P}_1 \subset A_0$.

Put $M_1 = A_0 \setminus \overline{P}_1$. Draw in M_1 a simple arc L_2 so that in the neighborhood of any point of the set M_1 of radius 10^{-4} there exists at least one point of L_2. Let $P_2 \supset L_2$ be such an open strip that $\mu_2(P_2) < 10^{-4}$ and $\overline{P}_2 \subset M_1$. Put $M_2 = M_1 \setminus \overline{P}_2$.

On the disk A define the function $\varphi(x, y)$ as follows: 1) $\varphi(x, y) = 0$ outside $\bigcup_{n=1}^{\infty} P_n$; 2) $\varphi(x, y) = 2^{-(n+1)}$ on L_n $(n = 1, 2, ...)$, and extend $\varphi(x, y)$ in each strip P_n continuously so that in P_n one has $0 \leq \varphi(x, y) \leq 2^{-(n+1)}$, and each level set of $\varphi(x, y)$ in the strip P_n consists of a single component. Let I be the graph of the function $\varphi(x, y)$, I_1 the graph of the function $\varphi(x, y) - 1/4$, and I_2 the graph of the function $\varphi(x, y) + 1/4$.

Define the function of three variables $F(\eta)$ as follows: 1) $F(\eta) = 0$ for $\eta \in I_1 \cup I_2 \cup (R \setminus Q_1)$, and also in the cylinder Q_2 between the plane $z = -1/2$ and the surface I_1, and between the surface I_2 and the plane $z = 1/2$; 2) $F(\eta) = 1$, if $\eta \in I$, and it is linear in z between the surfaces I_1 and I and between I and I_2; 3) in the cylindrical coordinate system φ, r, z, the function $F(\eta)$ is linear in r for $1/4 \leq r \leq 1/2$. $F(\eta)$ is continuous and the variations $W_0(F)$ and $W_1(F)$ are bounded. Let

$$W = \bigcap_{n=1}^{\infty} M_n \subset A_0.$$

Then $\mu_2(W) > 0$.

Denote by Ω the part of the cylinder over W, with generator parallel to the z-axis, enclosed between I_1 and I_2. Then $\mu_3(\Omega) > 0$, and at the points of Ω the function $F(\eta)$ is not differentiable.

Example 3. Let M be the sphere of radius 2^{-1} with center at O. We shall consider a partition of M into n^2 spherical polygons by a uniform net of n parallels and meridians. Divide M into 4 polygons M_1, M_2, M_3, M_4. In each M_i choose a spherical segment S_i so that the sum of the areas of these segments does not exceed $2\pi \cdot 10^{-8}$. Let $\sigma_0 = \bigcup S_i$. Divide M into polygons of the 1st rank so that the sum of the areas of the polygons of the 1st rank intersecting σ_0 does not exceed $4\pi \cdot 2^{-8}$, and the maximal diameter of these polygons is not greater than 10^{-1}.

Let $M_{1,1}, M_{1,2}, \dots, M_{1,k_1}$ be the spherical polygons of the 1st rank not intersecting σ_0. In each $M_{1,i}$ choose a segment $\sigma_{1,i}$ so that the sum of the areas of the segments $S_{1,i}$ does not exceed $2\pi \cdot 2^{-10}$. Let

$$P_1 = \bigcup_{i=1}^{k_1} M_{1,i} \quad \text{and} \quad \sigma_1 = \bigcup_{i=1}^{k_1} S_{1,i}.$$

Divide P_1 into polygons of the 2nd rank so that the sum of the areas of the polygons of the 2nd rank intersecting σ_1 does not exceed $4\pi \cdot 2^{-10}$, and the maximal diameter of a polygon of the 2nd rank does not exceed 10^{-2}. Continue this process indefinitely.

On each of the disks cutting off the segments of the systems $\sigma_0, \sigma_1, \dots$, as on bases, construct cones whose vertices are located outside the ball bounded by the sphere M, and the height of the cone supported on a segment from σ_l is equal to $2^{-(l+1)}$. Denote by I the surface thus obtained, and by I^* the open body bounded by the surface I. Define $F(\eta)$ as follows: 1) $F(\eta) = 0$, if $\eta \in R \setminus I^*$; 2) $F(\eta) = 1$ at the origin O; 3) $F(\eta)$ is linear on each ray emanating from O; 4) $F(\eta)$ is continuous, and the variations $W_0(F)$ and $W_2(F)$ are bounded. Let

$$W = M \setminus \bigcap_{n=1}^{\infty} P_n.$$

Then $\text{mes}_2 W > 0$. Denote by Ω the intersection of R and the cone over W with vertex at O. $F(\eta)$ is not differentiable at the points of Ω, and $\mu_3(\Omega) > 0$.

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Note: Figure translations are in progress. See original paper for figures.