ON GRAVITATIONAL PRESSURE IN FRIEDMANN-LOBACHEVSKY SPACE

PHYSICS

1970

SovietRxiv

Abstract

Full Text

UDC 530.12:531.51

PHYSICS

I. G. FIKHTENGOL’ TS

ON GRAVITATIONAL PRESSURE IN FRIEDMANN-LOBACHEVSKY SPACE

(Presented by Academician V. A. Fock on 22 IX 1969)

Consider a space whose fundamental tensor has the form

$$g_{\mu\nu} = \psi \eta_{\mu\nu}, \tag{1}$$

where

$$\psi = \psi(x_0, x_1, x_2, x_3) > 0,$$

$$\eta_{00} = 1, \qquad \eta_{0i} = 0, \qquad \eta_{ik} = -\delta_{ik}.$$

Here, as usual, $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$. Greek indices take the values $0, 1, 2, 3$, Latin indices the values $1, 2, 3$.

It follows from (1) that

$$g^{\mu\nu} = \frac{1}{\psi} \eta^{\mu\nu}, \tag{2}$$

where, of course,

$$\eta^{00} = 1, \qquad \eta^{0i} = 0, \qquad \eta^{ik} = -\delta_{ik}.$$

For the Christoffel symbols of the second kind we obtain, according to (1) and (2),

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \left(\delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + \delta^{\alpha}_{\nu} \delta^{\beta}_{\mu} - \eta_{\mu\nu} \eta^{\alpha\beta} \right) \frac{\partial \ln \psi}{\partial x_{\beta}}, \tag{3}$$

where, as usual, $\delta^\alpha_\beta = 1$ for $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ for $\alpha \neq \beta$. Summation over identical Greek indices is assumed from 0 to 3.

The Riemann tensor

$$R^\alpha_{\mu,\beta\nu} = \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x_\nu} - \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\beta} + \Gamma^\sigma_{\mu\beta}\Gamma^\alpha_{\nu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\beta\sigma} \tag{4}$$

can be represented in the form

$$R^\alpha_{\mu,\beta\nu} = P^\alpha_{\mu,\beta\nu} - \frac{1}{3}\left(\Lambda - \frac{R}{2}\right)\left(\delta^\alpha_\beta g_{\mu\nu} - \delta^\alpha_\nu g_{\mu\beta}\right). \tag{5}$$

Here, for the space under consideration,

$$P^\alpha_{\mu,\beta\nu} = \frac{1}{2}\left(\delta^\alpha_\beta P_{\mu\nu} - \delta^\alpha_\nu P_{\mu\beta}\right) + \frac{1}{2}\left(P^\alpha_\beta g_{\mu\nu} - P^\alpha_\nu g_{\mu\beta}\right) - \frac{P}{6}\left(\delta^\alpha_\beta g_{\mu\nu} - \delta^\alpha_\nu g_{\mu\beta}\right), \tag{6}$$

$$P_{\mu\nu} = \frac{\partial^2 \ln\psi}{\partial x_\mu \partial x_\nu} - \frac{1}{2}\frac{\partial \ln\psi}{\partial x_\mu}\frac{\partial \ln\psi}{\partial x_\nu}, \tag{7}$$

$$\Lambda = g^{\mu\nu}\left(\frac{\partial^2 \ln\psi}{\partial x_\mu \partial x_\nu} + \frac{1}{4}\frac{\partial \ln\psi}{\partial x_\mu}\frac{\partial \ln\psi}{\partial x_\nu}\right), \tag{8}$$

with

$$P^\nu_\mu = g^{\nu\sigma}P_{\mu\sigma}, \qquad P^\alpha_{\mu,\alpha\nu} = P_{\mu\nu}, \qquad P = g^{\mu\nu}P_{\mu\nu} \tag{9}$$

and, of course,

$$R^\alpha_{\mu,\alpha\nu} = R_{\mu\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}.$$

From equality (5) it follows directly that

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = P_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{10}$$

and, consequently, Einstein' s equations of gravitation can be represented in the form

$$P_{\mu\nu} - \Lambda g_{\mu\nu} = -\chi T_{\mu\nu}, \tag{11}$$

where $T_{\mu\nu}$ is the mass tensor and χ is Einstein' s gravitational constant.

Of course, representing the solution of Einstein' s equations of gravitation in the form (1) is possible only under certain assumptions concerning the mass tensor. In the present work we shall not consider the conditions that the mass tensor must satisfy in order for equations (11) to be mutually compatible if the metric has the form (1).

Put

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{12}$$

where ρ is the invariant mass density and u_μ is the four-dimensional covariant velocity vector, normalized by the formula

$$g^{\mu\nu} u_\mu u_\nu = 1. \tag{13}$$

The conditions

$$\nabla_\nu T^{\mu\nu} = 0, \tag{14}$$

where ∇_ν denotes the tensor derivative, are equivalent, as applied to (12), to the conditions

$$\rho u^\nu \nabla_\nu u^\mu + u^\mu \nabla_\nu \rho u^\nu = 0. \tag{15}$$

Here u^μ is the four-dimensional contravariant velocity vector.

Taking into account the continuity equations

$$\nabla_\nu \rho u^\nu = 0 \tag{16}$$

the conditions (14), as applied to (12), are in fact equivalent to the equations of a geodesic line

$$u^\nu \nabla_\nu u^\mu = 0. \tag{17}$$

If in the equations of a geodesic line the time coordinate x_0 is taken as the independent variable, then they are written in the form (see [1,2])

$$\ddot{x}_i = \dot{x}_i \Gamma^0_{\mu\nu} \dot{x}_\mu \dot{x}_\nu + \Gamma^i_{\mu\nu} \dot{x}_\mu \dot{x}_\nu = 0 \quad (i = 1, 2, 3). \tag{18}$$

Here the dot above denotes differentiation with respect to the time coordinate x_0.

According to (3)

$$\Gamma^{\alpha}_{\mu\nu}\dot{x}_\mu\dot{x}_\nu = \left(\dot{x}_\alpha\dot{x}_\beta - \frac{1}{2}\eta_{\mu\nu}\dot{x}_\mu\dot{x}_\nu\eta^{\alpha\beta}\right)\frac{\partial \ln\psi}{\partial x_\beta}, \tag{19}$$

and the equations of a geodesic line in the space under consideration take the form

$$\ddot{x}_i + \frac{1}{2}\left(\frac{\partial \ln\psi}{\partial x_i} + \frac{\partial \ln\psi}{\partial x_0}\dot{x}_i\right)\eta_{\mu\nu}\dot{x}_\mu\dot{x}_\nu = 0. \tag{20}$$

We satisfy equations (20) if we put

$$\psi = \psi(S), \tag{21}$$

where

$$S = \sqrt{\eta_{\mu\nu}x_\mu x_\nu}, \tag{22}$$

$$x = \dot{x}_i x_0, \tag{23}$$

where, by virtue of the relations (23) themselves,

$$\dot{x}_i = \text{const}. \tag{24}$$

With the aid of relations (23), as is known, the phenomenon of the "recession" of galaxies is explained, and the quantities $\dot{x}_i$ are regarded as the coordinates of the corresponding mass in the comoving coordinate system (see ([1,3–5])).

Applying (21)–(22),

$$\frac{\partial \ln\psi}{\partial x_\mu} = \frac{1}{S}\frac{d\ln\psi}{dS}\eta_{\mu\sigma}x_\sigma, \tag{25}$$

$$\frac{\partial^2 \ln\psi}{\partial x_\mu \partial x_\nu} = \frac{1}{S}\frac{d\ln\psi}{dS}\eta_{\mu\nu} + \frac{1}{S^2}\left(\frac{d^2\ln\psi}{dS^2} - \frac{1}{S}\frac{d\ln\psi}{dS}\right)\eta_{\mu\sigma}\eta_{\nu\tau}x_\sigma x_\tau, \tag{26}$$

and therefore, according to (7) and (8), we have, respectively,

$$P_{\mu\nu} = \frac{1}{S}\frac{d\ln\psi}{dS}\eta_{\mu\nu} + \frac{1}{S^2}\left\{\frac{d^2\ln\psi}{dS^2} - \frac{1}{S}\frac{d\ln\psi}{dS} - \frac{1}{2}\left(\frac{d\ln\psi}{dS}\right)^2\right\}\eta_{\mu\sigma}\eta_{\nu\tau}x_\sigma x_\tau, \tag{27}$$

$$\Lambda = \frac{1}{\psi}\left\{\frac{d^2\ln\psi}{dS^2} + \frac{3}{S}\frac{d\ln\psi}{dS} + \frac{1}{4}\left(\frac{d\ln\psi}{dS}\right)^2\right\}. \tag{28}$$

Taking into account (10), (27), and (28), we arrive at the conclusion that

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\left\{\frac{d^2\ln\psi}{dS^2} + \frac{2}{S}\frac{d\ln\psi}{dS} + \frac{1}{4}\left(\frac{d\ln\psi}{dS}\right)^2\right\}\eta_{\mu\nu} +$$

$$+\frac{1}{S^2}\left\{\frac{d^2\ln\psi}{dS^2} - \frac{1}{S}\frac{d\ln\psi}{dS} - \frac{1}{2}\left(\frac{d\ln\psi}{dS}\right)^2\right\}\eta_{\mu\sigma}\eta_{\nu\tau}x_\sigma x_\tau. \tag{29}$$

If, following (1), we set

$$\psi = H^2(S), \tag{30}$$

then equality (29) leads to an analogous equality obtained in (1) for the Friedman-Lobachevsky space. This space is considered in detail in (1) (see also (3-5)).

In (1) it is established that, as applied to (12), (29), and (30),

$$H = (1 - A/S)^2, \tag{31}$$

$$u_\mu = \frac{H}{S}\eta_{\mu\sigma}x_\sigma, \tag{32}$$

$$\rho = \frac{12A}{\varkappa S^3 H^3}, \tag{33}$$

where A is a constant.

Starting from equalities (28), (30), (31), and (33), we arrive at the expression for Λ

$$\Lambda = \frac{\varkappa}{c^2}p_I, \tag{34}$$

where

$$p_I = \frac{c^2\rho}{3}\sqrt{H}. \tag{35}$$

Equalities (10) and (34) give grounds to interpret the quantity p_I (35) as gravitational pressure (pressure of gravitational radiation) in the space under consideration.

According to (11),

$$P = 4\Lambda - \varkappa T, \tag{36}$$

where P is the invariant of the tensor $P_{\mu\nu}$, and T is the invariant of the mass tensor.

In accordance with (34),

$$P = \frac{4\varkappa}{c^2}\left(p_I - \frac{1}{4}c^2 T\right). \tag{37}$$

The quantity P (37) characterizes the curvature of the space under consideration, taking into account gravitational pressure.

What has been set forth gives grounds to interpret the tensor

$$P_{\mu\nu} = R_{\mu\nu} - {}^1/_2 g_{\mu\nu} R + \Lambda g_{\mu\nu} \tag{38}$$

as a tensor characterizing the curvature of space, taking gravitational pressure into account.

I take this opportunity to express my gratitude to Acad. V. A. Fock for discussing the work.

Leningrad Institute
of Precision Mechanics and Optics

Received
10 IX 1969

References

1. V. A. Fock, *Theory of Space, Time and Gravitation*, Moscow, 1961.
2. I. G. Fichtenholz, ZhETF, **36**, 1322 (1959).
3. L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, "Nauka," 1967.
4. Ya. B. Zel' dovich, I. D. Novikov, *Relativistic Astrophysics*, "Nauka," 1967.
5. A. Friedmann, *Zs. Phys.*, **10**, 377 (1922).