ON SCALES OF GROWTH OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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ON SCALES OF GROWTH OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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Below we use the following notation: R^n is n-dimensional Euclidean space; $r = (r_1, \dots, r_n)$, $u = (u_1, \dots, u_n)$, etc.; $R^n_+ = \{r \in R^n : r_i \geq 0\}$, $R^n_0 = \{r \in R^n : r_i > 0\}$; $\varphi(r) = \varphi(r_1, \dots, r_n)$; $W(e^u) = W(e^{u_1}, \dots, e^{u_n})$; $\varphi(r^\gamma) = \varphi(r_1^{\gamma_1}, \dots, r_n^{\gamma_n})$;

$$|r| = \left(\sum_1^n r_i^2\right)^{1/2}; \qquad \|r\| = \sum_{i=1}^n r_i; \qquad (k, u) = \sum_{i=1}^n k_i u_i;$$

$(C, V(u))$ is a pair consisting of a function $V(u)$ and a set C, where $V(u)$ is defined and finite; $\{(u, u_{n+1}) \in R^{n+1} : u \in C,\ u_{n+1} \geq V(u)\} = [R^n, V(u)]$ is the epigraph of the function $V(u)$.

1°. The growth of an entire function $f(z) = f(z_1, \dots, z_n)$ of n complex variables $z_1, \dots, z_n$ is very often compared with the growth of its majorant

$$M_f(r) = \max_{|z_i| \leq r_i} |f(z)|.$$

Typical for investigation is the class $\mathfrak{M}_n = \{f(z)\}$ of entire functions transcendental in at least one variable:

$$\mathfrak{M}_n = \{f(z) : 0 < \gamma_f \overset{\text{def}}{=} \overline{\lim_{t \to +\infty}} (\ln t)^{-1} \ln \ln M(t, \dots, t) < +\infty\}.$$

For $n = 1$, $\mathfrak{P}_1 = \{\exp\{r^\gamma\},\ \gamma > 0\}$ is the widely known scale of growth of functions of the class $\mathfrak{M}_1$. For each function $f(z) \in \mathfrak{M}_1$ in the scale $\mathfrak{P}_1$ there exists a unique function $\exp\{r^{\gamma_f}\}$ asymptotically equivalent to $M_f(r)$ in the known sense, and the quantity γ_f is then called the order of growth of the function $f(z)$.

The theory of growth of entire functions of several variables, over its more than half-century history, beginning with the work of E. Borel ([1]), has accumulated many analogues of the scale $\mathfrak{P}_1$ and corresponding analogues of the notion of order of an entire function of one variable. As a scale of growth of functions of the class $\mathfrak{M}_n$, functions of the form $\{\exp\{[\varphi(r)]^\gamma\},\ \gamma > 0\}$ are chosen, where $\varphi(r)$ is a function of relatively simple structure, connected with a definite method of exhausting R^n_+ (for example, for E. Borel [1], $\varphi(r) = \max\{r_1, \dots, r_n\}$; for P. Lelong [2], $\varphi(r) = |r|$), or systems of such functions (for example, for L. I. Ronkin [3], $\gamma = 1$, $\varphi(r) = r_1^{\gamma_1} + \dots + r_n^{\gamma_n}$, $\gamma_i > 0$). Here results of a very general character are due to A. A. Gol' dberg [4].

Let G be a closed bounded domain in $R^n_\perp$. By the (G, x)-order of an entire function $f(r)$ with respect to the collection of variables we shall call the quantity

$$\rho_G(x) = \overline{\lim_{t\to+\infty}} (\ln t)^{-1} \ln\ln M(t), \quad \text{where } M(t) = \sup_{c\in G} M_f(t^{x_1 c_1}, \dots, t^{x_{nc} n}),$$

and any element of the set

$$S_f = \{\gamma \in R^n : \rho_G(\gamma_1^{-1}, \dots, \gamma_n^{-1}) = 1\} \tag{1}$$

will be called a system of conjugate G-orders of the function $f(z)$ (cf. [3]). Recall that $(R^n_0, \rho_G(x))$ does not depend on G: $\rho_G(x) \equiv \rho(x)$ [4]. Suppose additionally that $G \in \mathfrak{B}$, where $\mathfrak{B}$ is the class of closed bounded complete logarithmically convex domains in R^n_+. Then the above definitions lead to scales of growth of the function $M_f(r)$, consisting of some-

functions in R^n_+, nondecreasing in each of the variables $r_1, \dots, r_n$, convex with respect to $\ln r_1, \dots, \ln r_n$. It is precisely these properties that the function $\ln M_f(r)$ possesses [5].

In the present paper we investigate the growth scales of A. A. Gol' dberg. Starting from this analysis and developing results on the asymptotics of convex functions in [6], we propose qualitatively new growth scales for functions of the class $\mathfrak{M}_n$. These scales completely take into account certain asymptotic properties of the function $M_f(r)$.

2°. Let $H = \{\varphi(r)\}$ be the class of functions in R^n such that: 1) $\varphi(r) > 0$ for $r \in R^n_+ \setminus 0$; 2) $\varphi(\lambda r) = \lambda\varphi(r)$ for $\forall\lambda > 0,\ r \in R^n_+$; 3) $\varphi(r)$ is nondecreasing in each variable; 4) $\varphi(r)$ is continuous in R^n_+; 5) $\varphi(r)$ is convex with respect to $\ln r_1, \dots, \ln r_n$ for $r \in R^n_0$; $W_f(r) = \ln\ln M^+_f(r)$, $M^+_f(r) = \max\{M_f(r), e\}$ *.

Theorem 1. *Let x be an arbitrary element of R^n_0. In order that the number $a \geq 0$ be the G-order of an entire function $f(z)$ for some $G \in \mathfrak{B}$, it is necessary and sufficient that there exist a function $\varphi(r) \in H$ such that*

$$a = \lim_{\|r\|\to+\infty} W_f(r)[\ln\varphi(r^\gamma)]^{-1}, \tag{2}$$

where $\gamma_i = x_i^{-1}$, $i = 1, \dots, n$, and

$$G = \{r \in R_+^n : \ \varphi(r^\gamma) \le 1\}.$$

The definitions of orders according to A. A. Gol' dberg therefore lead to the following system of growth scales for functions of the class $\mathfrak{M}_n$: $E = \{E(\varphi;\gamma),\ \varphi \in H,\ \gamma \in R_0^n\}$, where $E(\varphi;\gamma) = \{\exp[(\varphi(r^\gamma))^\tau],\ \tau > 0\}$, and for every function $f(z)$ from $\mathfrak{M}_n$ there exists, in each scale $E(\varphi;\gamma)$, a unique function

$$a(r;\gamma) = \exp\{[\varphi(r^\gamma)]^{\rho(x)}\},$$

where $x_i = \gamma_i^{-1},\ i = 1,\dots,n$; $\rho(x)$ is the x-order of the function $f(z)$, asymptotically equivalent to $M_f(r)$ in the following sense:

$$\lim_{\|r\|\to+\infty} W_f(r)\cdot[\ln\ln a(r;\gamma)]^{-1} = 1.$$

3°. In studying the asymptotics of the growth of functions of the class $\mathfrak{M}_n$, a natural object of investigation is the quasiconvex function ([7])

$$V(u) = W_f(e^u)$$

(i.e.,

$$V(\lambda u + \mu v) \le \max\{V(u), V(v)\}, \quad \forall \lambda + \mu = 1;\ \lambda,\mu > 0;\ u, v \in R^n).$$

Definition 1. Let $\Omega_V = \{K\}$ be the collection of cones with vertex at $O \in R^{n+1}$, some shifts of which belong to the epigraph $[R^n, V(u)]$ of the quasiconvex function $(R^n, V(u))$. The **asymptotic cone** $\Pi(V)$ of the epigraph $[R^n, V(u)]$ is the set

$$\overline{\bigcup_{K\in\Omega_V} K}$$

**.

Let us introduce the following characteristic of growth ([6], p. 584):

Definition 2. The **function of growth orders** of an entire function $f(z)$ is the function $(D, \rho_f(u))$, where

$$\rho_f(u) = \lim_{t\to+\infty} W_f(e^{u_1 t},\dots,e^{u_n t})\,t^{-1}, \qquad D = \{u \in R^n :\ \rho_f(u) < +\infty\}.$$

1) If $f \in \mathfrak{M}_n$, then $D = R^n$, and $\Pi(W_f) = [R^n, \rho_f(u)]$, where $\Pi(W_f)$ is the asymptotic cone of the epigraph of the function $\widetilde{W}_f(e^u)$.
2) For every $\varepsilon > 0$ there exists a number $C_\varepsilon > 0$ such that

$$\ln M_f(e^u) < C_\varepsilon \exp\{\rho_f(u) + \varepsilon|u|\}, \qquad \forall u \in R^n.$$

Thus the function $(R^n, \rho_f(u))$ completely determines the cone $\Pi(W_f)$, takes into account the growth of $M_f(r)$ in all directions (for each fixed $u \in R^n$, $\rho_f(u)$ is the order of growth of the function $\Phi_u(r) = M_f(r^{u_1}, \dots, r^{u_n})$); with the aid of $\rho_f(u)$ a simple global upper estimate of the function $M_f(r)$ is possible,

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* In studying the asymptotics of the growth of $M_f(r)$, it is enough to restrict oneself to its truncation from below.

** Simple examples show that it is not always the case that $\Pi(V) \in \Omega_V$. For convex functions this definition passes over into the previously known one ([8]).

i.e., $\rho_f(u)$ is the growth characteristic of the function $f(z)$. If $n = 1$, then $\rho_f(u) = \max\{0, \rho u\}$, where ρ is the order of growth of $f(z)$, i.e., the cone $\Pi(W_f)$ is determined by one number—an element of R_0^1. For $n > 1$ this is not true in the general case. The cone $\Pi(W_f)$ is determined by the directrix T_f of its surface, situated in the horizontal hyperplane $u_{n+1} = 1$: $T_f = \{u \in R^n : \rho_f(u) = 1\}$. The hypersurface S_f of conjugate orders (formula (1)) determines only part of the cone $\Pi(W_f) \cap R^{n+1}+$, since $S_f = T_f^- \cap R_0^n$, where T_f^- is the hypersurface obtained from T_f by the transformation $y_i = u_i^{-1}$, $i = 1, \dots, n$.

4°. There is a simple connection between the asymptotic cone $\Pi(W_f)$ of the epigraph of the function $W_f(e^u)$, $f \in \mathfrak{M}_n$, and the asymptotic cone $\Pi(\Theta_\gamma^+)$ for $\forall\, \gamma \in R_0^n$, where $\Theta_\gamma^+(u) = \max\{\ln\ln \alpha(r;\gamma), 0\}$ (see 2°).

Proposition 1. *The cone $\Pi(\Theta_\gamma^+)$ is the intersection of a finite number of half-spaces, and $\Pi(\Theta_\gamma^+) \subset \Pi(W_f)$ for $\forall\, \gamma \in R_0^n$, and the cone $\Pi(\Theta_\gamma^+)$ touches the surface of the cone $\Pi(W_f)$ along the ray*

$$\{(t\gamma_1^{-1}, \dots, t\gamma_n^{-1}, t\rho_f(\gamma_1^{-1}, \dots, \gamma_n^{-1})),\ t \geq 0\}.$$

If $n > 1$, then from the preceding and from the results of L. I. Ronkin ([3]), the author ([9]) concludes that the directrix of the cones $\{\Pi(W_f) \cap R_+^{n+1},\ f \in \mathfrak{M}_n\}$, taken in the hyperplane $u_{n+1} = 1$, may be any closed convex complete domain in R_+^n. Therefore the structure of the asymptotic cones of the epigraphs $\{[R^n, W_f(e^u)],\ f \in \mathfrak{M}_n\}$ is much more complicated than that of the epigraphs of functions of systems $\{\ln\ln \psi(e^u),\ \psi \in E(\varphi;\gamma)\},\ \varphi \in H,\ \gamma \in R_0^n$.

5°. Let Y be the class of functions in R^n that are nonnegative, convex, nondecreasing in each variable, and positively homogeneous (of degree 1). In addition, assume that the function $\varphi_0(u) \equiv 0 \notin Y$. Consider the following scale of growth Q_n of functions of the class $\mathfrak{M}_n$: $Q_n = \{\exp(\exp(\tilde{\varphi}(r))),\ \varphi \in Y\}$, where $\tilde{\varphi}(r)$ is the continuous extension to R_+^n of the function $(R_0^n, \varphi(\ln r_1, \dots, \ln r_n))$. For $n = 1$, $Q_1 = \{\exp\{\max\{r^\gamma, 1\}\},\ \gamma > 0\}$ (cf. with $\mathfrak{B}_1, 1°$).

The asymptotic properties of the order functions $\rho_f(u)$, $f \in \mathfrak{M}_n$, are clarified by

Theorem 2. *If $f(z)$ is an arbitrary function from $\mathfrak{M}_n$, then*

$$\overline{\lim}_{|u|\to+\infty,\ u\in K} W_f(e^u) \cdot [\rho_f(u)]^{-1} = 1 \tag{3}$$

for any cone K with vertex at O and such that $\overline{K} \setminus \{0\} \subset \{u \in R^n : \rho_f(u) > 0\}$, and $\exp(\exp(\mathfrak{F}_f(r)))$ is the unique function of the scale Q_n asymptotically equivalent to $M_f(r)$ in the sense of condition (3).

Theorem 3. *For any function $\varphi(u)$ of the class Y there exists an entire function $f(z) \in \mathfrak{M}_n$ such that $\rho_f(u) \equiv \varphi(u)$.*

The desired function is, for example,

$$f(z) = \sum_{s=0}^{+\infty} \sum_{\|k\|=s} \exp\{-p_s(k)\ln s\}\, z_1^{k_1} \ldots z_n^{k_n}.$$

Here $p_s(u)$ is the Minkowski functional ([10]) of the set $O_{s^{-1/2}}(K_\varphi)$; $O_\varepsilon(M)$ is the ε-neighborhood of the set M; K_φ is the convex compact set whose support function is $\varphi(u)$. If $\dim K_\varphi = n$, then there also exists a simpler desired function:

$$f(z) = \sum_{k\in C} \exp\{p(k)\ln\|k\|\}\, z_1^{k_1} \ldots z_n^{k_n}.$$

Here $p(u)$ is the Minkowski functional of K_φ; C is the smallest convex cone with vertex at O containing K_φ. (Note that $O \in K_\varphi \subset R^n_+$.) The idea of the example just given was suggested by the results of G. Valiron ([5]) and

* The example of the function $(z_1 + z_2)e^{z_1 z_2}$ shows that this requirement is essential for the fulfillment of condition (3).

L. I. Ronkina ([3]). In the proof of Theorem 3, the following result, which is of independent interest, is used.

Theorem 4. For all $u \in R^n$, $\rho_f(u) \equiv \tau_f(u)$, $\forall f \in \mathfrak{M}_n$, where

$$\tau_f(u) = \begin{cases} \overline{\lim}_{(k,u)\to+\infty} \dfrac{(k,u)\ln(k,u)}{-\ln|a_k|}, & u \in A, \\ 0, & u \in R^n \setminus A, \end{cases}$$

$\{a_k = a_{k_1\ldots k_n};\ k_1, \ldots, k_n = 0, 1, \ldots\}$ are the Taylor coefficients of the function $f(z)$,

$$A = \bigcap_{m>0} A_m, \qquad A_m = \{u \in R^n : \{k : a_k \neq 0\} \cap \{k : (k,u) > m\} \neq \emptyset\}.$$

Theorem 4 is a generalization of a well-known result of A. A. Goldberg ([4]).

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Note: Figure translations are in progress. See original paper for figures.