A REFLECTED ONE-DIMENSIONAL RAREFACTION WAVE IN A CONSTANT GRAVITATIONAL FIELD

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Abstract

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A REFLECTED ONE-DIMENSIONAL RAREFACTION WAVE IN A CONSTANT GRAVITATIONAL FIELD

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In the one-dimensional outflow of the products of an instantaneous explosion, situated in a constant gravitational field and obeying the equation of state $P = \text{const} \cdot \rho^k$, a rarefaction wave will travel to the left into the gas at rest; the solution of the equation for it was obtained for an arbitrary adiabat k in general form in work ([1]). The aim of the present article is to determine the motion that arises when the rarefaction wave is reflected from a rigid wall.

For the indicated flow the equations of gas dynamics, written in the form

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2}{k-1}\,c\frac{\partial c}{\partial x} &= -a, \\ \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + \frac{k-1}{2}\,c\frac{\partial u}{\partial x} &= 0, \end{aligned} \tag{1}$$

have the general solution ([1])

$$\psi = \frac{\partial^{n-1}}{\partial i^{n-1}} \frac{F_1\left[\sqrt{2(2n+1)i} + w\right] + F_2\left[\sqrt{2(2n+1)i} - w\right]}{\sqrt{i}},$$

$$x = x_0 + wt - at^2/2 - \partial\psi/\partial w, \qquad t = \partial\psi/\partial i, \tag{2}$$

$$x_0 = -n(2n+1)c_{\text{H}}^2/a, \qquad k = (2n+3)/(2n+1), \qquad n = 0, 1, 2, \ldots$$

Here c is the speed of sound in the given medium; c_{H} is the initial speed of sound in the section $x = 0$; $i = c^2/(k-1)$ is the heat content of the gas; $w = u + at$, where a is the acceleration of gravity and u is the flow velocity along the x-axis.

Let the rarefaction wave arise at the instant $t = 0$ in the section $x = 0$ and propagate in the negative direction of the x-axis. Since the compressed gas in the gravitational field before outflow was in a state of adiabatic equilibrium,

$$c\frac{\partial c}{\partial x} = -\frac{k-1}{2}\,a, \qquad u = 0$$

or

$$c^2 = c_{\mathrm{H}}^2 - (k-1)ax, \qquad u = 0, \tag{3}$$

then the rarefaction wave, propagating according to the law $dx/dt = -c$, will reach a rigid wall placed in the section $x = -l$ after the interval

$$t = t_l = \frac{2}{(k-1)a}\left[\sqrt{c_{\mathrm{H}}^2 + (k-1)al} - c_{\mathrm{H}}\right] \tag{4}$$

and will be reflected from it. In this reflected wave the function F_2 is retained:

$$F_2 = F_2[\omega - (w - \omega_{\mathrm{H}})] = \frac{1}{4a\,(n+1)!\,[2(2n+1)]^{\,n+1/2}}\times$$

$$\times\frac{\partial^{\,n-1}}{\partial i^{\,n-1}}\frac{\sum\limits_{r=n+1}^{2n} A_r[\omega - (w + \omega_{\mathrm{H}})]^r}{\sqrt{i}}. \tag{5}$$

This is obvious, since the given function is determined by the condition

$$w = \frac{2}{k-1}(c_{\mathrm{n}} - c) = \sqrt{2(2n+1)i_{\mathrm{n}}} - \sqrt{2(2n+1)i} = \omega_{\mathrm{n}} - \omega$$

on the right characteristic $dx/dt = u + c$ and has a constant value on the left characteristic.

The function $F_1 = F_1[\sqrt{2(2n+1)i} + w] = F_1(\omega + w)$ must be found from the condition that $u \equiv 0$ at $x = -l$. Since F_1 depends on arguments that are nothing other than the characteristic condition

$$w + \frac{2}{k-1}c = \mathrm{const},$$

where the constant at the time $t = t_l$ is equal to

$$\mathrm{const} = \frac{2}{k-1}\left[2\sqrt{c_{\mathrm{n}}^2 + (k-1)al} - c_{\mathrm{n}}\right] = \frac{2}{k-1}c_{\mathrm{n}}^*,$$

the condition

$$F_1 = \text{const} \quad \text{for} \quad w + \frac{2}{k-1}c = \frac{2}{k-1}c_{\text{n}}^*$$

must be satisfied.

We seek the solution for ψ in the form

$$\psi = -\frac{1}{4a(n+1)![2(2n+1)]^{n+1/2}} \left\{ \frac{\partial^{n-1}}{\partial i^{n-1}} \frac{\sum_{r=n+1}^{2n} A_r(\omega - w - \omega_{\text{n}})^r}{\sqrt{i}} + \frac{\partial^{n-1}}{\partial i^{n-1}} \frac{F_1(\omega + w)}{\sqrt{i}} \right\}, \tag{6}$$

where the coefficients A_r are computed algebraically for each n (1).

In finding the explicit form of the function F_1 for an arbitrary value of n, computational difficulties arise. Below we consider the special case $n = 0$, for which it is easy to find the explicit form of the function F_1, which will make it possible to construct the function F_1 for other values of n. For the indicated case $F_2 = 0$, as is seen from (6), and $\psi = F_1(\omega + w)$.

From (2)

$$x = ut + at^2/2 - F_1', \qquad ct = F_1'.$$

For $x = -l$, $u \equiv 0$, $w = at$, $w + c = at + c = z$, whence

$$t = (z - c)/a, \qquad F_1' = c(z - c)/a \tag{7}$$

or

$$aF_1' = (c - z + z)(z - c) = -(z - c)^2 + z(z - c),$$

whence

$$z - c = \frac{1}{2}\left(z \pm \sqrt{z^2 - 4aF_1'}\right).$$

Using the equality

$$l = F_1' - a^2t^2/2a = F_1' - (z - c)^2/2a,$$

for F_1' we obtain the expression

$$F_1' = \frac{3}{2}l + \frac{z^2}{9a} - \frac{z^2}{9a}\sqrt{z^2 - 6al}$$

or

$$\frac{\partial\psi}{\partial w} = \frac{2}{3}l + \frac{(w+c)^2}{9a} - \frac{(w+c)}{9a}\sqrt{(w+c)^2 - 6al}. \tag{8}$$

Thus, knowledge of the derivative $\partial\psi/\partial w$ determines the solution for the reflected wave.

It should be noted that this solution (for $k = 3$) coincides with the result previously obtained by K. P. Stanyukovich by another method (the solution was "guessed") (2):

$$\sqrt{2ax+\alpha^2} + (at+\alpha) = 2\sqrt{2ax+\alpha^2+2al},$$

where $\alpha = u + c$, which after an elementary transformation can be written as

$$x = \frac{5}{9}(w+c)t - \frac{5}{18}at^2 - \frac{2}{9a}(w+c)^2 - \frac{4}{3}l + \frac{2(w+c)}{9a}\sqrt{(w+c)^2 - 6al}.$$

We obtain this same result if, in the expression $x = wt - at^2/2 - \partial\psi/\partial w$, the derivative $\partial\psi/\partial w$ is replaced according to (8), which confirms the correctness of the approach in choosing the function F_1.

For the case $n = 1$ ($k = 5/3$—a monatomic gas)

$$i = 3c^2/2, \qquad \omega = 3c = \sqrt{6i},$$

and solution (2) may be written in the form

$$\psi = \frac{\omega_{\mathrm{H}}^2}{12a\omega}\left[\omega - (w+\omega_{\mathrm{H}})\right]^2 + \frac{F_1(\omega+w)}{a\omega}, \tag{9}$$

$$x = x_0 + ut + \frac{at^2}{2} - \partial\psi/\partial w, \qquad t = \partial\psi/\partial i = \partial\psi/3c\,\partial c = 3\,\partial\psi/\omega\,\partial\omega,$$

or, for x and t, we have the expressions

$$x = x_0 + ut + \frac{at^2}{2} + \frac{\omega_{\mathrm{H}}^2}{6a\omega}\left[\omega - (w+\omega_{\mathrm{H}})\right] - \frac{F_1'}{a\omega},$$

$$t = -\frac{\omega_{\mathrm{H}}^2}{4a\omega^3}\left[\omega - (w+\omega_{\mathrm{H}})\right]^2 + \frac{\omega_{\mathrm{H}}^2}{2a\omega^2}\left[\omega - (w+\omega_{\mathrm{H}})\right] + \frac{3F_1'}{a\omega^2} - \frac{3F_1}{a\omega^3}.$$

Hence, under the condition that $u = 0,\ w = at$, at $x = -l$ we have

$$-l = x_0 + \frac{at^2}{2} + \frac{\omega_{\text{H}}^2}{6a\omega}\left[\omega - (at + \omega_{\text{H}})\right] + \frac{F_1'}{u\omega}, \tag{10}$$

$$t = -\frac{\omega_{\text{H}}^2}{4a\omega^3}\left[\omega - (at + \omega_{\text{H}})\right]^2 + \frac{\omega_{\text{H}}^2}{2a\omega^2}\left[\omega - (at + \omega_{\text{H}})\right] + \frac{3F_1'}{a\omega^2} - \frac{3F_1'}{a\omega^3}. \tag{11}$$

If we introduce a new variable

$$z = w + \omega = at + \omega,$$

then

$$F_1 = F_1(w + \omega) = F_1(\omega + at) = F_1(z),$$

$$at = at + \omega - \omega = z - \omega, \qquad \omega - (at + \omega_{\text{H}}) = 2\omega - \omega_{\text{H}} - z,$$

and relations (10) and (11) will take, respectively, the form

$$-3(l + x_0)a = \frac{3(z - \omega)^2}{2} + \frac{\omega_{\text{H}}^2}{2\omega}(2\omega - \omega_{\text{H}} - z) + \frac{2F_1'}{\omega},$$

$$\omega(z - \omega) = -\frac{\omega_{\text{H}}^2}{4\omega^2}(2\omega - \omega_{\text{H}} - z)^2 + \frac{\omega_{\text{H}}^2}{2\omega}(2\omega - \omega_{\text{H}} - z) + \frac{3F_1'}{\omega} - \frac{3F_1'}{\omega^2}. \tag{12}$$

We transform the system of equations (12). The first equation gives

$$F_1' = \frac{\omega_{\text{H}}^2}{6}(\omega_{\text{H}} + z) - \omega\left[\frac{\omega_{\text{H}}^2}{3} + \frac{z^2}{2} + a(l + x_0)\right] + \frac{z\omega^2}{2} - \frac{\omega^3}{2}.$$

Hence, solving the cubic equation, we find $\omega = \omega(F_1'; z)$; substituting $\omega = \omega(F_1'; z)$ into the second equation, we arrive at the equation

$$\Phi(F_1'; F_1; z) = 0, \tag{13}$$

solving which we formally find $F_1 = F_1(z)$, which completely solves the posed problem.

It is obvious that already for $k = 5/3$ ($n = 1$) the problem is algebraically complicated, but can nevertheless be carried through to the end. For $n = 2, 3, 4, \ldots$ this becomes impossible, since the equations contain ω in a degree higher than the fourth. Therefore the problem must be solved approximately, approximating the isentrope $P = A_0\rho^k$ by the isentrope $P = A\rho^3 - B$; in this case the accuracy will be sufficient, since in the reflected wave the pressure does not depend very strongly on the coordinate x.

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REFERENCES

1. K. P. Stanyukovich, K. Dzhusupov, *DAN*, **177**, No. 4 (1967).

2. K. P. Stanyukovich, *Unsteady Motions of a Continuous Medium*, § 76, Moscow, 1955.

Note: Figure translations are in progress. See original paper for figures.