ON THE ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR A SYSTEM OF QUASILINEAR EQUATIONS WITH THE SAME PRINCIPAL PART CONTAINING A SMALL PARAMETER

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Abstract

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ON THE ASYMPTOTICS OF THE SOLUTION OF A PROBLEM WITH AN INITIAL JUMP FOR A SYSTEM OF QUASILINEAR EQUATIONS WITH THE SAME PRINCIPAL PART CONTAINING A SMALL PARAMETER

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§ 1. Consider the system of differential equations

$$\varepsilon(\partial u/\partial t + \partial u/\partial x) = F(t, x, u, v),$$
$$\partial v/\partial t + \partial v/\partial x = G(t, x, u, v), \tag{1}$$

with initial conditions

$$\xi(\varepsilon)u(t, x)\big|_{t=0} = u_0^0(x), \quad v(t, x)\big|_{t=0} = v_0^0(x), \tag{2}$$

where $\varepsilon > 0$ is a small parameter, and $\xi(\varepsilon)$ is a certain function of ε, which will be determined below. Suppose that the right-hand sides F and G of system (1) have the representations:

$$F(t, x, u, v) = u^n \left[f(t, x, v) + \sum_i{}' f_i(t, x, v)u^{-i} \right],$$
$$G(t, x, u, v) = u^m \left[g(t, x, v) + \sum_i{}' g_i(t, x, v)u^{-i} \right], \tag{3}$$

where n and m are certain real numbers satisfying the conditions:
a) $m = n,\ n \geq 1$; b) $n - 1 < m < n,\ n \geq 1$; c) $m = n - 1,\ n > 1$, and in the strip $Q(0 \leq t \leq T < \infty,\ -\infty < x < +\infty)$, for all values of v,

$$f(t, x, v) < 0, \quad g(t, x, v) > 0. \tag{4}$$

Then the solution of problem (1), (2), with the corresponding choice of the function $\xi(\varepsilon)$, will, as $\varepsilon \to 0$, tend to the solution of the degenerate system

$$0 = F(t,x,u,v),$$
$$\partial v/\partial t + \partial v/\partial x = G(t,x,u,v), \qquad (5)$$

but, however, this solution will not satisfy the initial condition $v(t,x)\big|_{t=0} = v_0^0(x)$, but will satisfy another initial condition:

$$v(t,x)\big|_{t=0} = v_0^0(x) + \Delta(x), \qquad (6)$$

where $\Delta(x)$ will be called the initial jump of the function $v(t,x)$. The existence of an initial jump of the function $v(t,x)$ depends on the form of the function $\xi(\varepsilon)$ (the form of $\xi(\varepsilon)$, in turn, depends essentially on the character of the nonlinearity of the right-hand side of system (1)), namely

$$\xi(\varepsilon) = \begin{cases} \varepsilon, & m = n,\ n \geq 1; \\ \varepsilon^{1/(m+1-n)}, & n-1 < m < n,\ n \geq 1; \\ e^{1/\varepsilon}, & m = n-1,\ n > 1. \end{cases} \qquad (7)$$

In the present note we investigate case b) for $n > 1$, while the remaining cases are investigated analogously. The initial jump $\Delta(x)$, by virtue of (4), is deter-

is determined uniquely from the equation

$$\left(u_0^0(x)\right)^{m+1-n} \ \| \ -(m+1-n)$$

To construct the asymptotics of the solution of problem (1), (2), we divide the strip Q into three strips ([1], [2]):

$$Q_1(0 \leq t \leq t_1^0,\ -\infty < x < +\infty), \qquad t_1^0 = O\left(\varepsilon^{[m-\sigma(n-1)]/(m+1-n)}\right),$$
$$Q_2(t_1^0 \leq t \leq t_2^0,\ -\infty < x < +\infty), \qquad t_2^0 = O(\varepsilon);$$
$$Q_3(t_2^0 \leq t \leq T,\ -\infty < x < +\infty), \qquad 0 < \sigma < 1.$$

§ 2. **Construction of the asymptotics in the first strip.** We make the change of variables

$$\tau = t/\varepsilon^{\alpha}, \qquad \varepsilon^{\beta} u = z, \qquad \alpha = [m - \sigma(n-1)]/(m+1-n),$$
$$\beta = (1-\sigma)/(m+1-n).$$

Then the first strip Q_1 is transformed into the strip $Q_1^0(0 \leq \tau \leq \tau_1^0,\ -\infty < x < +\infty)$, where $\tau_1^0 = O(1)$ as $\varepsilon \to 0$, and problem (1), (2) takes the form

$$\frac{\partial z}{\partial \tau} + \varepsilon^{\alpha}\frac{\partial z}{\partial x} = z^n \left[f(\varepsilon^{\alpha}\tau, x, v) + \sum_i{}' \varepsilon^{i\beta} f_i(\varepsilon^{\alpha}\tau, x, v) z^{-i} \right],$$

$$\frac{\partial v}{\partial \tau} = \varepsilon^{\alpha}\frac{\partial v}{\partial x} = \varepsilon^{\sigma} z^m \left[g(\varepsilon^{\alpha}\tau, x, v) + \sum_i{}' \varepsilon^{i\beta} g_i(\varepsilon^{\alpha}\tau, x, v) z^{-i} \right], \tag{9}$$

$$\varepsilon^{\sigma/(m+1-n)} z(\tau, x, \varepsilon)\big|_{\tau=0} = u_0^0(x), \qquad v(\tau, x, \varepsilon)\big|_{\tau=0} = v_0^0(x). \tag{10}$$

We seek an approximate solution of problem (9) and (10) in the form

$$z(\tau, x, \varepsilon) = z_0(\tau, x, \varepsilon) + \sum_{s,p}{}' \varepsilon^{s\alpha+p\beta} z_{sp}(\tau, x, \varepsilon),$$

$$v(\tau, x, \varepsilon) = v_0(\tau, x, \varepsilon) + \sum_{s,p}{}' \varepsilon^{s\alpha+p\beta} v_{sp}(\tau, x, \varepsilon). \tag{11}$$

Substituting (11) into (9), in the first approximation we obtain

$$\partial z_0/\partial \tau = f(0, x, v_0) z_0^n, \qquad \varepsilon^{\sigma/(m+1-n)} z_0(\tau, x, \varepsilon)\big|_{\tau=0} = u_0^0(x),$$

$$\partial v_0/\partial \tau = \varepsilon^{\sigma} g(0, x, v_0) z_0^m, \qquad v_0(\tau, x, \varepsilon)\big|_{\tau=0} = v_0^0(x). \tag{12}$$

Solving this system and using formula (8) for the initial jump, for $z_0(\tau, x, \varepsilon)$ and $v_0(\tau, x, \varepsilon)$ we obtain the estimates

$$\overline{\Delta}(x) - v_0(\tau, x, \varepsilon) = O(\varepsilon^{\sigma}) \left(\tau + \varepsilon^{\sigma(n-1)/(m+1-n)}\right)^{-(m+1-n)/(n-1)},$$

$$z_0(\tau, x, \varepsilon) = O(1) \left(\tau + \varepsilon^{\sigma(n-1)/(m+1-n)}\right)^{-1/(n-1)}, \tag{13}$$

where $\overline{\Delta}(x) \equiv v_0^0(x) + \Delta(x)$. Hence, at $\tau = \tau_1^0$ one can directly verify that

$$z_0(\tau_1^0, x, \varepsilon) = O(1), \qquad \overline{\Delta}(x) - v_0(\tau_1^0, x, \varepsilon) = O(\varepsilon^{\sigma}). \tag{14}$$

For $z_{sp}(\tau, x, \varepsilon)$, $v_{sp}(\tau, x, \varepsilon)$ we have the problems:

$$\partial z_{sp}/\partial \tau = n z_0^{n-1} f(0, x, v_0) z_{sp} + z_0^n f_v'(0, x, v_0) v_{sp} + \Phi_{sp},$$

$$\partial v_{sp}/\partial \tau = \varepsilon^{\sigma}\left[m z_0^{m-1} g(0, x, v_0) z_{sp} + z_0^m g_v'(0, x, v_0) v_{sp}\right] + \Psi_{sp}, \tag{15}$$

$$z_{sp}(\tau, x, \varepsilon)\big|_{\tau=0} = 0, \qquad v_{sp}(\tau, x, \varepsilon)\big|_{\tau=0} = 0.$$

The estimates hold

$$z_{sp}(\tau,x,\varepsilon) = O(z_0(\tau,x,\varepsilon))\,\frac{1}{(z_0(\tau,x,\varepsilon))^{s(n-1)+p}},$$

$$v_{sp}(\tau,x,\varepsilon) = O(1)\,\frac{1}{(z_0(\tau,x,\varepsilon))^{s(n-1)+p}}. \tag{16}$$

Under the appropriate conditions on F, G, $u_0^0(x)$, and $v_0^0(x)$, the existence and uniqueness of the solution of problem (1), (2) follows from [3].

Theorem 1. *Every solution u, v of problem (1), (2) in Q_1 admits the asymptotic expansion*

$$u = \varepsilon^{-(1-\sigma)/(m+1-n)}\left[z_0(\tau,x,\varepsilon) + \sum_{s+p=1}^{N}\varepsilon^{s\alpha+p\beta}z_{sp}(\tau,x,\varepsilon)\right] + R_N,$$

$$v = v_0(\tau,x,\varepsilon) + \sum_{s+p=1}^{N}\varepsilon^{s\alpha+p\beta}v_{sp}(\tau,x,\varepsilon) + S_N, \tag{17}$$

and for R_N and S_N in the closed domain Ω_1, bounded by the straight lines $t=0$, $t=t_1^0$ and the characteristics of system (1), the estimates uniform in t, x hold

$$R_N = O\left(\sum_{s+p=N+1}\varepsilon^{(sm+p-1)/(m+1-n)}\right), \qquad S_N = O\left(\sum_{s+p=N+1}\varepsilon^{(sm+p)/(m+1-n)}\right). \tag{18}$$

§ 3. Construction of the asymptotics in the second strip

In the strip Q_2, system (1) is solved with the initial conditions

$$\varepsilon^{(1-\sigma)/(m+1-n)}u(t,x,\varepsilon)\big|_{t=t_1^0} = z_1^0(x,\varepsilon), \qquad v(t,x,\varepsilon)\big|_{t=t_1^0} = v_1^0(x,\varepsilon), \tag{19}$$

where $z_1^0(x,\varepsilon)$ and $v_1^0(x,\varepsilon)$ are the exact values of the solution at $t=t_1^0$, bounded as $\varepsilon\to 0$. Make the change $t = t_1^0+\varepsilon\tau$; then the second strip Q_2 passes into the strip Q_2^0 $(0\le\tau\le\tau_2^0,\ -\infty<x<+\infty)$, where $\tau_2^0 = O(1)$ as $\varepsilon\to 0$, and problem (1), (19) takes the form:

$$\partial u/\partial\tau + \varepsilon\,\partial u/\partial x = F(t_1^0+\varepsilon\tau,x,u,v),$$

$$\partial v/\partial\tau + \varepsilon\,\partial v/\partial x = \varepsilon G(t_1^0+\varepsilon\tau,x,u,v), \tag{20}$$

$$\varepsilon^{(1-\sigma)/(m+1-n)} u(\tau, x, \varepsilon)\big|_{\tau=0} = z_1^0(x, \varepsilon), \qquad v(\tau, x, \varepsilon)\big|_{\tau=0} = v_1^0(x, \varepsilon).$$

We seek an approximate solution of problem (20) in the form

$$u(\tau, x, \varepsilon) = u_0(\tau, x, \varepsilon) + \varepsilon u_1(\tau, x, \varepsilon) + \cdots,$$

$$v(\tau, x, \varepsilon) = v_0(\tau, x, \varepsilon) + \varepsilon v_1(\tau, x, \varepsilon) + \cdots. \tag{21}$$

Substituting (21) into (20), in the first approximation we obtain the equation for $u_0(\tau, x, \varepsilon)$

$$\partial u_0/\partial \tau = F(t_1^0, x, u_0, v_1^0(x, \varepsilon)), \qquad \varepsilon^{(1-\sigma)/(m+1-n)} u_0\big|_{\tau=0} = z_1^0(x, \varepsilon). \tag{22}$$

Assuming now that $F = O(u_0^n)$, from (22) we obtain

$$u_0(\tau, x, \varepsilon) = O\left[\left(\tau + \varepsilon^{(n-1)(1-\sigma)/(m+1-n)}\right)^{-1/(n-1)}\right]. \tag{23}$$

Hence we note that, for $\tau = \tau_2^0$, the function $u_0(\tau_2^0, x, \varepsilon) = O(1)$ as $\varepsilon \to 0$.

Substituting (21) into (20) and comparing the coefficients of equal powers of ε, we obtain a system of equations for determining the coefficients.

$u_k(\tau, x, \varepsilon), v_k(\tau, x, \varepsilon)$ of the expansions (21):

$$\begin{gathered} \partial u_k/\partial \tau = F'_u(t_1^0, x, u_0, v_0) u_k + F'_v(t_1^0, x, u_0, v_0) v_k + \Phi_k, \\ \partial v_k/\partial \tau + \Psi_k, \qquad u_k\big|_{\tau=0} = 0, \quad v_k\big|_{\tau=0} = 0. \end{gathered} \tag{24}$$

The estimates hold

$$\begin{gathered} u_k(\tau, x, \varepsilon) = O\left[(u_0(0, x, \varepsilon))^{k(m+1-n)}\right] u_0(\tau, x, \varepsilon), \\ v_k(\tau, x, \varepsilon) = O\left[(u_0(0, x, \varepsilon))^{k(m+1-n)}\right], \end{gathered} \tag{25}$$

where $u_0(\tau, x, \varepsilon)$ is the solution of equation (22).

Theorem 2. Every solution u, v of problem (1), (19) in the second strip Q_2 admits the asymptotic expansion

$$\begin{aligned} u(t, x, \varepsilon) &= \sum_{k=0}^{N} \varepsilon^k u_k\left(\frac{t - t_1^0}{\varepsilon}, x, \varepsilon\right) + R'_N, \\ v(t, x, \varepsilon) &= \sum_{k=0}^{N} \varepsilon^k v_k\left(\frac{t - t_1^0}{\varepsilon}, x, \varepsilon\right) + S'_N, \end{aligned} \tag{26}$$

and for R'_N and S'_N in the domain Ω_2, bounded by the straight lines $t = t_1^0$, $t = t_2^0$ and by the characteristics of system (1), the estimates uniform in t, x hold

$$R'_N = O\big(\varepsilon^{(N+1)\sigma-(1-\sigma)/(m+1-n)}\big)\,, \qquad S'_N = O\big(\varepsilon^{(N+1)\sigma}\big)\,. \tag{27}$$

We shall not dwell here on the construction of the asymptotics of the solution in the third strip. We only note that it can be constructed by using the results of works ([4],[5]).

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[1] M. I. Vishik, L. A. Lyusternik, *Dokl. Akad. Nauk SSSR*, **132**, No. 6 (1960).
[2] K. A. Kasymov, a) *Dokl. Akad. Nauk SSSR*, **179**, No. 2 (1968); b) collection *Equations of Mathematical Physics and Functional Analysis*, Alma-Ata, 1966.
[3] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 1952.
[4] V. A. Trenogin, *Tr. Moscow Phys.-Tech. Inst., Studies in Mechanics and Applied Mathematics*, issue 9, 1962.
[5] K. A. Kasymov, *Izv. Acad. Sci. Kazakh SSR*, issue 3 (1970).

Note: Figure translations are in progress. See original paper for figures.