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Abstract

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ON THE FORMULATION OF EQUATIONS FOR SCATTERING WITH REARRANGEMENT IN THE COORDINATE REPRESENTATION

In the present article we reproduce, in the coordinate representation, the derivation of nonsingular integral equations for the problem of scattering with rearrangement involving three composite fragments. The relations obtained make it possible, in developing the considerations of ([1]), to elucidate the difficulties arising in attempts to use the adiabatic approximation in a problem with rearrangement, indicate the necessity of restricting the choice of effective potentials in the distorted-wave method, and also provide the possibility of a simple way of obtaining general threshold formulas. At the same time, owing to the use of the coordinate representation, all the arguments acquire a very transparent character.

Let us first consider scattering with rearrangement in a system of three fragments composed of different particles, at energies for which only the reaction channels are open that correspond to the union of any two fragments into a bound state. We choose the zero of energy in the center-of-mass system so that the indicated condition is expressed as $E \leqslant 0$. We number the fragments by the index $\alpha = 1, 2, 3$ and denote their masses by m_α. The reaction channels under consideration will also be denoted by the index α, it being understood that in the reaction channel α the fragment α goes to infinity, while the two other fragments remain bound. In each channel we introduce Jacobi coordinates $\mathbf{R}_\alpha, \mathbf{r}^\alpha(\xi_\alpha)$, where $\mathbf{R}_\alpha$ is the distance between the center of gravity of fragment α and the common center of gravity of the two remaining fragments; $\mathbf{r}_\alpha$ is the distance between the centers of gravity of these two fragments, and (ξ_α) are the remaining "internal" coordinates, which in what follows do not enter explicitly and will not be written out. We denote the Hamiltonian of the system by $\mathcal{H} = \mathcal{K} + V = \mathcal{K} + V_\alpha + v_\alpha$, where $\mathcal{K}$ is the kinetic-energy operator, $\mathcal{K} = \mathbf{P}_\alpha^2/2M_\alpha + \mathbf{p}_\alpha^2/2m_\alpha$ ($\alpha = 1, 2, 3$), and v_α denotes that part of the interaction which tends (by assumption sufficiently rapidly) to zero as $R_\alpha \to \infty$. Here $\mathbf{P}_\alpha$ and $\mathbf{p}_\alpha$ are the momentum operators corresponding to motion in the coordinates $\mathbf{R}_\alpha$ and $\mathbf{r}_\alpha$, $M_\alpha = m_\alpha(m_\beta + m_\gamma)/M$, $m_\alpha = m_\beta m_\gamma/(m_\beta + m_\gamma)$,

$M = m_1 + m_2 + m_3$ $(\alpha \neq \beta, \gamma; \gamma \neq \alpha, \beta)$. We shall seek the complete wave function of the scattering problem Ψ in the form of the sum $\Psi = \Psi_1 + \Psi_2 + \Psi_3$, where each of the functions Ψ_α has the nonzero asymptotic form required in the scattering problem only in its "own" channel and zero asymptotic form in the remaining channels, and satisfies the system of equations

$$\begin{aligned}(\mathcal{K} + v_1 - E)\Psi_1 &= -V_1(\Psi_2 + \Psi_3),\\ (\mathcal{K} + v_2 - E)\Psi_2 &= -V_2(\Psi_3 + \Psi_1),\\ (\mathcal{K} + v_3 - E)\Psi_3 &= -V_3(\Psi_1 + \Psi_2).\end{aligned} \tag{1}$$

The separation of the function Ψ into parts is equivalent to a rearrangement of the Lippmann–Schwinger equation, carried out by L. D. Faddeev ([2]), and was performed earlier by Sasakawa ([3]). We shall show the possibility of a noncontradictory construction of the functions Ψ_α with the indicated properties directly in the coordinate (and not the usually used momentum) representation. In this case system (1) possesses the fundamental advantage in comparison ···

to the Schrödinger equation for Ψ, that the sources entering in (1) are localized in the space of the variables $\mathbf{R}_\alpha, \mathbf{r}_\alpha$ in all directions, which makes it possible to pass directly from (1) to "good" integral equations by means of the Green functions G_α $(\alpha = 1, 2, 3)$, which in their "own" coordinate representation have the form

$$\langle \mathbf{R}_\alpha, \mathbf{r}_\alpha | G_\alpha | \mathbf{R}_\alpha, \mathbf{r}_\alpha \rangle =$$

$$= i \sum_{LMlmn_\alpha} \int_{E_{0\alpha}}^{\infty} d\mathcal{E}_\alpha \left\{ \frac{1}{k_\alpha R_\alpha} Y_{LM}\left(\frac{\mathbf{R}_\alpha}{R_\alpha}\right) \varphi_\alpha(lmn_\alpha \mathcal{E}_\alpha; \mathbf{r}_\alpha) \left[\theta(R_\alpha - R'_\alpha) h^{(1)}_{L+1/2}(k_\alpha R_\alpha)\right.\right.$$

$$\left. \times j_{L+1/2}(k_\alpha R_\alpha) + \theta(R'_\alpha - R_\alpha) h^{(1)}_{L+1/2}(k_\alpha R'_\alpha) j_{L+1/2}(k_\alpha R_\alpha)\right] \cdot \frac{1}{R'_\alpha} Y_{LM}$$

$$\left. \times \left(\frac{\mathbf{R}'_\alpha}{R'_\alpha}\right) \varphi_\alpha(lmn_\alpha \mathcal{E}_\alpha; \mathbf{r}'_\alpha) \right\}, \tag{2}$$

$$k_\alpha = \sqrt{2M_\alpha(E - \mathcal{E}_\alpha) + i\varepsilon}, \qquad \varepsilon \to +0,$$

where $h^{(1)}_{L+1/2}$ and $j_{L+1/2}$ are Riccati-Bessel functions, with $h^{(1)}_{L+1/2}(k_\alpha R_\alpha)$ for $R_\alpha \to \infty$ containing only outgoing or decaying waves; Y_{LM} are spherical harmonics normalized to unit solid angle, and $\varphi_\alpha(lmn_\alpha \mathcal{E}_\alpha; \mathbf{r}_\alpha)$ are components of a complete set of solutions of the equation

$$(p_\alpha^2/2m_\alpha + V_\alpha - \mathcal{E}_\alpha)\varphi_\alpha = 0, \tag{3}$$

where $\mathcal{E}_\alpha$ runs through the whole spectrum of values of the two-particle problem,* beginning with the assumed existing lowest value $\mathcal{E}_{0\alpha}$; l, m are the total intrinsic angular momentum and the projection of the angular momentum, and n_α are the remaining discrete quantum numbers of the fragments. In doing so we use the results of ([4]) concerning the same value of the threshold energy of breakup in all reaction channels. The system (1), together with the boundary conditions of the scattering problem, can be represented in the form of a matrix integral equation with a completely continuous kernel coinciding with the kernel of the Faddeev equations:

$$\Psi_i = \Psi_{0i} = G^0 V \Psi_i, \tag{4}$$

where

$$\Psi_i = \begin{vmatrix} \Psi_{1i} \\ \Psi_{2i} \\ \Psi_{3i} \end{vmatrix}, \qquad \mathbf{G}_0 = \begin{vmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{vmatrix}, \qquad \mathbf{V} = \begin{vmatrix} 0 & V_1 & V_1 \\ V_2 & 0 & V_2 \\ V_3 & V_3 & 0 \end{vmatrix}, \tag{5}$$

the index i corresponds to a fixed initial state, which is described by a 3×1 matrix (column) containing, depending on the formulation of the problem, the product of an incident plane or spherical wave by the corresponding bound state in the row corresponding to the initial channel, and zero in the remaining rows. The solution of equation (4) indeed satisfies the necessary conditions, since for $R_\alpha \to \infty$ the second term on the right-hand side contains nondecaying spherical waves only in channel α. This follows from the fact that each of the Green functions making up the matrix G_0 is multiplied by a localized source, and therefore, in the limit considered (for $R_\alpha \to \infty$), the terms in (2) containing $\theta(R'_\alpha - R_\alpha)$ vanish. In each of the "foreign" channels the functions Ψ_α tend asymptotically to zero, since in this case r_α and R_α tend simultaneously to infinity. In doing so one must keep in mind that on the physical sheet a positive $E - \mathcal{E}_\alpha$ will correspond in the Riccati-Bessel functions to an approach to the cut from above, while for

* The integral over $d\mathcal{E}_\alpha$ in (2) includes summation over the discrete spectrum and integration over the contin

$E - \mathcal{E}_\alpha < 0$ both $h^{(1)}_{L+1/2}$ and $j_{L+1/2}$ behave as $e^{-|\kappa_\alpha| R_\alpha}$. It is important to emphasize that the above considerations concerning the localization of Ψ_α in its own channels are valid only for $E \leq 0$; for $E > 0$ the entire argument must evidently be substantially modified, and, possibly, a transition is necessary, as in ([2,3]), to the momentum representation, since the interval of integration $0 \leq \mathcal{E}_\alpha \leq E$ in the Green functions (2), after their substitution into (4), will give nondecaying waves corresponding to the channel of breakup of the system into three fragments and "overlapping" with all the remaining channels.

Considering the asymptotics of the right-hand side of (4) after the substitution (2), we find that the scattering amplitude into the state $f = \{l^f, m_x^f, n_\alpha^f, L_\alpha^f, M_\alpha^f, \mathcal{E}_\alpha\}$ of channel α, determined by the coefficient in front of the corresponding asymptotic outgoing spherical wave, is given in the coordinate representation by the expression

$$A_{fi} = \int \Psi_{0f}^*(\mathbf{R}_\alpha, \mathbf{r}_\alpha^f)\, V_{\alpha f}(\mathbf{r}_\alpha^f)\, (\Psi_{i\beta} + \Psi_{i\gamma})\, d^3\mathbf{R}_\alpha d^3\mathbf{r}_\alpha, \tag{6}$$

$$\Psi_{0f}(\mathbf{R}_\alpha, \mathbf{r}_\alpha) = \frac{1}{k_\alpha R_\alpha}\, Y_{LM}\left(\frac{\mathbf{R}_\alpha}{R_\alpha}\right) j_{L+1/2}(k_\alpha R_\alpha)\, \varphi_\alpha(l_\alpha, m_\alpha, n_\alpha, \mathcal{E}_\alpha; r_\alpha)$$

or, in matrix notation,

$$A_{fi} = (\Psi_{0f},\, \mathbf{V}\Psi_i), \tag{7}$$

where the brackets denote the scalar product, including integration over all coordinates, and Ψ_{0f} is a column with a single nonzero row corresponding to the final channel.

Let us introduce the symbolic solution of (4), writing it in the form of the column Ψ_i,

$$\Psi_i = \Psi_{0i} - \mathbf{G}\mathbf{V}\Psi_{0i}, \tag{8}$$

where $\mathbf{G}$ is the solution of an equation with a "good" kernel (owing to the absence of diagonal elements in the matrix V),

$$\mathbf{G} = \mathbf{G}_0 - \mathbf{G}_0\mathbf{V}\mathbf{G}. \tag{9}$$

With the aid of (8), the amplitude A_{fi} is represented as follows:

$$A_{fi} = (\Psi_{0f},\, (\mathbf{V} - \mathbf{V}\mathbf{G}\mathbf{V})\Psi_{0i}). \tag{10}$$

We emphasize that in deriving (10) we nowhere used the condition of orthogonality of the wave functions of different channels. It is also interesting to note that (10) contains terms entering the Gell-Mann-Goldberger formula ([5]) in a rearranged form. If one speaks in the language of Weinberg diagrams ([6]), then the last (left) interaction in the notation of the expressions for the amplitude is the interaction responsible for the formation of the bound state.

All the preceding considerations can be simply extended to the case of a larger number of fragments, as well as to the case of the presence of identical particles in different fragments. In the latter case the number of reaction channels must be increased in accordance with the possible number of permutations of identical particles between the fragments, and the number of equations in the system (1) increases accordingly. Subsequent symmetrization makes it possible to restore the number of equations, but the new system will already contain non-local interactions involving the permutation operator. A construction similar to that carried out above is evidently possible only under a definite separation of interactions in the different channels. Thus this approach, as well as the rearrangement of L. D. Faddeev in general, does not allow, as was noted in ([1]), an expansion of the functions Ψ in terms of so-called adiabatic functions, i.e., eigenfunctions of the full Hamiltonian from which one or another part of the operator has been excluded.

kinetic energy. The latter expansion is especially often used in atomic-molecular problems, without the necessary reduction of the resulting equations to a system with a "good" kernel.

The splitting of the interactions on which the writing of equations (1) is based is not unique. For example, to V_1 we may add any sufficiently effective interaction, decreasing with distance, of the form $W(R_1)P_1$, where P_1 is the projection operator onto the eigenfunctions (3) corresponding to the discrete spectrum in channel 1, and, correspondingly, subtract this interaction from V_2 and (or) V_3. In this case the interaction $W(R_1)$ must not cause rearrangement and, for example, must depend only on R_1. The entire treatment remains unchanged; only the functions from which, in the coordinate representation, the Green function is constructed are changed. Introduction of the operator P_1 is mandatory in order to preserve the local character of the sources in (1). In a similar way one can obtain formulae analogous to the formulae of the distorted-wave method, and carry out a generalized threshold analysis that takes into account the long-range part of the potential. This will be done in a subsequent work.

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REFERENCES

[1] A. M. Brodskii, V. G. Levich, DAN, 191, No. 1, 126 (1970).
[2] L. D. Faddeev, ZhETF, 39, 1459 (1960); Proc. Steklov Math. Inst., Acad. Sci. USSR, 69 (1963).
[3] T. Sasakawa, Phys. Rev., 158, 1249 (1967).
[4] W. Hunziker, Helv. phys. acta, 39, 451 (1966).
[5] M. Goldberger, K. Watson, *Collision Theory*, Moscow, 1967.
[6] S. Weinberg, Phys. Rev., 133, B 232 (1964).

Note: Figure translations are in progress. See original paper for figures.