PONTRYAGIN SPACE AND CONVERGENCE OF THE BUBNOV-GALERKIN METHOD

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Abstract

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A. and V. GERSH

PONTRYAGIN SPACE AND CONVERGENCE OF THE BUBNOV-GALERKIN METHOD

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Let H be a separable complex Hilbert space, and let A be a general self-adjoint operator. In this note we prove the convergence of the Bubnov-Galerkin method for the equation $Af = g$ under the assumption that the spectrum of A on the negative real semiaxis consists of no more than a finite number of eigenvalues of finite multiplicity and that the equation has a unique solution.

We shall proceed from the canonical decomposition of the operator A, $A = J|A|$. In the separable Hilbert space $H_{|A|}$ constructed, according to K. Friedrichs ([1]), by means of the positive operator $|A|$, we additionally introduce an indefinite metric determined by the use of the involutory operator J. Then in the J-space ([2]) $H_{|A|,J}$ obtained in this way, the Bubnov-Galerkin method will be equivalent to a process of J-orthogonalization, which, when all approximations exist, leads to a J-orthonormal system, and otherwise to a J-orthogonal system of subspaces. At the same time, an answer to the question of convergence of this method with respect to the metric of the Hilbert space $H_{|A|}$ can be obtained if it is known under what conditions such an orthogonalization process in $H_{|A|,J}$ leads to a Schauder basis or to a basis of subspaces [3].

The problem will be trivial for the special case $J = 1$, i.e., for the case of a positive definite operator A, for which the Bubnov-Galerkin method is identical with the Ritz method and all approximations exist, since every complete orthonormal system is a Schauder basis. For $J \neq 1$ the solution can be regarded as satisfactory only for J-spaces from the class of Pontryagin spaces ([4]).

The membership of the space $H_{|A|,J}$ in the class of these spaces is guaranteed in the present case by the assumptions made at the outset concerning the spectrum of the operator A. For the case when this operator has a bounded inverse, convergence of the method in $H_{|A|,J}$ will, in conclusion, imply convergence in H.

As is known from the investigations of R. Courant ([5]) and M. G. Krein ([6,7]), in the field of ordinary differential equations and partial differential equations, as well as integral equations, the class of operators established at the outset will have substantial realizations to which these arguments will be directly applicable, as will be shown by the example of an important class of ordinary differential operators. At the same time, the convergence assertions obtained in this way in connection with equations of the above-mentioned type, on the basis of the results of the present note, should not essentially go beyond the scope of the convergence proofs obtained by M. V. Keldysh ([8]) and S. G. Mikhlin ([9,10]), partly by direct investigations. However, on the basis of applying the theory of J-spaces to problems connected with the Bubnov–Galerkin method, one may suppose that the convergence of this method under simplifying assumptions and with the use of known basis-

criteria ([11, 3]) can also be shown for more general classes of operators. These questions will be considered in the following note.

1. First of all, we shall prove a necessary and sufficient criterion for the convergence of the Bubnov–Galerkin method for the special case of equations with involutory operators, i.e., with self-adjoint operators satisfying the relation $J^2 = 1$ and, consequently, possessing a spectral decomposition of the form $J = P^+ - P^-$, where the inertia indices of the operator J are determined by the dimensions of both projection operators. Equations of this kind are, obviously, trivial.

In view of the fact that the two Hilbert spaces mentioned at the beginning will, in this connection, be identical, in what follows only the space H_J will be relevant, as well as, along with J-orthonormalized systems $((J\psi_i, \psi_j) = \varepsilon_i \delta_{ij}\ (\varepsilon_i = \pm 1;\ i, j = 1, 2, ...))$, positive subspaces $((Jf, f) \geqslant 0;\ f \neq 0\ ((Jf, f) > 0))$, uniformly positive subspaces $((Jf, f) \geqslant \mu \|f\|^2,\ \mu > 0)$, maximal subspaces possessing these properties, and negative subspaces, defined analogously, of this space. The connection of J-orthonormalized systems with the representations of N. Bari ([12]), which we regard as known, was studied in greater detail by Yu. P. Ginzburg and I. S. Iokhvidov ([13]).

Lemma 1. Let $L \subset H_J$ be a positive subspace, and let $\{\psi_i\}$ be a complete in L J-orthonormalized system.

The following assertions are equivalent: a) L is uniformly positive; b) $\{\psi_i\}$ is a Riesz basis in L.

Lemma 2. Let $P \subset H_J$ be a maximal positive subspace, N the maximal negative complementary subspace belonging to it, and let $\{\chi_i\}_1^\infty$ be a complete in H_J J-orthonormalized system consisting of J-orthonormalized systems of these spaces. If $\{\chi_i\}_1^\infty$ has the unconditional basis property, then $H_J = P \dot{+} N$; moreover, P is uniformly positive.

The proof of Lemma 2 is essentially a consequence of Orlicz' s theorem ([14]).

Let now $Jf = g$ be of the type indicated at the beginning, and let $\{\varphi_i\}_1^\infty$ be a linearly independent system of elements, complete in H.

When the expression

$$f^n = \sum_{i=1}^{n} c_i^{(n)} \varphi_i$$

is used, the Bubnov–Galerkin equations will be

$$\sum_{i=1}^{n} c_i^{(n)} (J\varphi_i, \varphi_j) = (g, \varphi_j) \quad (j = 1, 2, \dots, n)$$

or

$$(Jf^{(n)}, \varphi_j) = (Jf, \varphi_j) \quad (j = 1, 2, \dots, n).$$

Theorem 1. The Bubnov–Galerkin method converges in H for the equation $Jf = g$ for any right-hand side and any complete linearly independent system of elements if and only if one of the two inertia indices of J is finite, i.e., if the space H_J belongs to the class of Pontryagin spaces.

Using the results of I. S. Iokhvidov and M. G. Krein ([15], [16]), we shall first show the convergence of the Bubnov–Galerkin method under the assumption that H_J belongs to the class of Pontryagin spaces, while the case $\dim P^- H_J = \chi < \infty$ takes place. With regard to the solvability of the Bubnov–Galerkin equations, it should be noted that they will always be solvable starting from a certain n^*. Namely, the J-metric may degenerate

* This conclusion will not be valid in the case of general J-spaces, as may be concluded from an example give

the largest one for a finite number of subspaces from the sequence of subspaces spanned by the first elements of the system $\{\varphi_i\}_1^\infty$.

This follows from the fact that one can choose such a canonical decomposition of the space H_J into a J-orthogonal sum $H^+ \dotplus H^-$ ($\dim H^- = \varkappa < \infty$), in which the space H^- will be contained in the linear span of the system $\{\varphi_i\}_1^\infty$. Thus there is a smallest (from the above sequence) finite-dimensional subspace M containing H^-. The assertion made applies to this and to the subsequent subspaces of the sequence. Using the Bubnov-Galerkin method, we first obtain a decomposition of the space H_J into the orthogonal sum $L \dotplus M$, where L ($\dim L = \infty$) is uniformly positive and its intersection with the above linear span is dense there. Then the method leads to the construction of a J-orthonormal system in L. The convergence being proved will be a consequence of Lemma 1.

It remains to show that from the convergence of the Bubnov-Galerkin method for each right-hand side of the equation and each complete linearly independent system of elements there follows that the space H_J belongs to the class of Pontryagin spaces. To this end we consider an arbitrary maximal positive subspace P of H_J, an additional subspace N, and a system, J-orthonormal and complete in H_J, according to Lemma 2. For the given method it suffices, under an arbitrary arrangement of the elements, to take this J-orthonormal system as the basis at once. Now convergence implies the unconditional basis property for this system and, according to Lemma 2, uniform positivity for the subspace P. In the rest the conclusion is a consequence of the result obtained by Yu. P. Ginzburg ([18], [13]), according to which a J-space ($\dim P^+H_J = \infty$), where all maximal positive and, consequently, all positive subspaces are uniformly positive, belongs to the type under consideration.

2. The dependences presented in § 1 are carried over (with the aid of the notion, studied by A. Wintner ([19]) and T. Kato ([20]), of the affinity of operators) to equations of the form $Af = g$, where A is a self-adjoint and entire operator, i.e. everywhere defined and continuous in both directions. According to ([19], [20]), every entire self-adjoint operator A will be affine to the involutory operator J, i.e. there exists an entire operator P and a representation $A = P^*JP$. Therefore the equation under consideration and the corresponding Bubnov–Galerkin equations can be reduced to the case studied in § 1 by the substitution $u = Pf,\ v = P^{*-1}g$. The inertia indices will now be invariants under an affine transformation. Therefore, also for this class of operators, a necessary and sufficient sign of convergence of the Bubnov–Galerkin method is the condition that one of the inertia indices of the operator A be finite.

If the operator A, as was assumed in the introduction, is only self-adjoint and invertible, then (taking into account the further result from ([20]), concerning the affinity of general self-adjoint operators to diagonal operators, or the connection that exists, owing to the commutativity of the operators $|A|$, A, and J, between the spectral decompositions) the transfer and generalization of these considerations to $H_{|A|,J}$ leads to a criterion of convergence of a sufficient character, since the domain D_A of definition of the operator A in general does not coincide with the space $H_{|A|,J}$. Necessity will exist only when elements of $H_{|A|J}$ not yet contained in D_A are taken into account as generalized solutions of the equation.

Thus we have

Theorem 2. *The Bubnov–Galerkin method converges in $H_{|A|}$ for the equation $Af = g$ (A a self-adjoint invertible operator) for every admissible right-hand side and every complete linearly independent system of elements in $H_{|A|}$ if and only if the space $H_{|A|J}$ belongs to the class*

Pontryagin space. This occurs in the case when the spectrum of the operator A on the negative (positive) real half-axis consists of at most a finite number of eigenvalues of finite multiplicity. Under this assumption the method converges in H, if the operator A has a bounded inverse.

Remark. The class of operators for which the assertions of Theorem 2 hold includes, in particular, self-adjoint extensions of positive operators with finite defect index (m, m). This remark is an immediate consequence of a known result from ([6]), according to which the negative part of the spectra of such extensions consists only of a finite number of eigenvalues, and the sum of their multiplicities does not exceed m.

This property of the spectrum, which is decisive for the applicability of Theorem 2, is possessed, according to ([7,21]), ultimately by those ordinary self-adjoint differential operators which in L^2 $(a \le x \le b)$ are obtained by extension under the boundary conditions

$$y^{[k]}\big|_{x=a} - y^{[k]}\big|_{x=b} = 0 \quad (k = 1, 2, \dots, 2n-1)$$

of the regular differential expression

$$y^{[2n]}(y^{[k]} = y^{(k)}((k) \text{ is the } k\text{-th derivative with respect to } x:\ k = 1, 2, \dots, n{-}1); \quad y^{[n]} = p_0 y^{(n)}, \quad y^{[n+k]} = p_n y^{(}$$

$$(k = 1, 2, \dots, n;\ y^{[0]} = y;\ 1/p_0, p_1, \dots, p_n \text{ summable on } (a, b);\ p_0 > 0).$$

Institute for Materials Testing
West Berlin

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Note: Figure translations are in progress. See original paper for figures.