A PRIORI ESTIMATES FOR EQUATIONS WITH A PARAMETER

MATHEMATICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.9

MATHEMATICS

A. S. BRATUS'

A PRIORI ESTIMATES FOR EQUATIONS WITH A PARAMETER

(Presented by Academician I. G. Petrovskii, 16 XII 1969)

Consider differential operators depending on a parameter, with smooth complex-valued coefficients in a bounded domain $\Omega \subset R_n$, of the form

$$P(x, D, q) = \sum_{|\alpha|+j\le m} a_{\alpha j}(x) q^j D^\alpha, \qquad x \in \Omega, \tag{1}$$

$$D^\alpha = \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Here q is a parameter taking values on the half-line Θ issuing from the origin of the complex plane. By $P_m(x, D, q)$ we denote the principal part of the operator P of order m.

In paper $(^1)$, boundary-value problems for operators of this type were investigated under the condition that the characteristic polynomial $P_m(x, \xi, q) \neq 0$ for $|\xi| + |q| \neq 0$, $\xi \in R_n$, $q \in \Theta$. In the present note the question is considered of the existence and regularity of solutions of the differential equation $P(x, D, q)u = f$ in the case when this condition is violated. The central point is the proof of estimates of the form

$$|q| \sum_{|\alpha|\le m-1} |q|^{2(m-1-|\alpha|)} \iint |D^\alpha u|^2\, dx \le K \iint |P(x, D, q)u|^2\, dx, \tag{2}$$

$$u \in C_0^\infty(\Omega), \qquad |q| \ge d_0 > 0.$$

Estimates of this type for operators not depending on a parameter, in the spaces $L_2(\Omega)$ with a certain weight, were considered in works $(^{2-5})$.

Theorem 1. *Let P be an operator of the form (1) with coefficients from $C^1(\overline{\Omega})$. Suppose that there exist a positive constant K and a number $d_0 > 0$ such that estimate (2) holds.*

Then, for those $(x, \xi_0, q_0) \in \overline{\Omega} \times R_n \times \Theta$ *for which* $\xi_0 \neq 0$, $q_0 \neq 0$, *and* $P_m(x, \xi_0, q_0) = 0$, *the inequality holds*

$$|\xi_0|^{2(m-1)} \leq 2K|q_0|^{-1} \operatorname{Im} \sum_{j=1}^{n} P_{m(j)}(x, \xi_0, q_0)\, \overline{P_m^{(j)}(x, \xi_0, q_0)}. \tag{3}$$

Here and below

$$P_{m(j)}(x, \xi, q) = \frac{\partial}{\partial x_j} P_m(x, \xi, q), \qquad P_m^{(j)}(x, \xi, q) = \frac{\partial}{\partial \xi_j} P_m(x, \xi, q).$$

Proof. Choose $w(x) \in C^\infty$ so that $w(x) = \langle x, \xi_0 \rangle + O(|x|^2)$ as $x \to 0$, and set $q = \lambda q_0$, where $\lambda > 0$. Consider the function $u^\lambda(x) = \exp[i\lambda w(x)]\psi(x\sqrt{\lambda})$, where $\psi \in C_0^\infty(R_n)$. Applying the Leibniz formula, introducing the new variable $x\sqrt{\lambda}$, and passing to the limit as $\lambda \to \infty$, we obtain that (2) is equivalent to

$$|\xi_0|^{2(m-1)}|q_0| \iint |\psi|^2\, dx \leq K \iint \left| \sum_{j=1}^{n} (P_{m(j)}(0, \xi_0, q_0) \cdot x_j)\psi + \sum_{j=0}^{n} P_m^{(j)}(0, \xi_0, q_0) D^j \psi \right|^2 dx;$$

using the result of papers (2), § 8, 1 and (3), § 1, 2, we obtain condition (3). The theorem is proved.

We now show that a condition of type (3) is also sufficient for the validity of estimate (2). Let us first consider the case of operators with real coefficients.

Theorem 2. *Let* P *be an operator of the form (1) with coefficients of the principal part from* $C^2(\overline{\Omega})$, *and let* $\operatorname{Im} q \neq 0$. *Suppose that at those points* $x \in \overline{\Omega}$ *where there exist such* $0 \neq \xi \in R_n$, $0 \neq q \in \Theta$, *that* $P_m(x, \xi, q) = 0$, *the condition*

$$\operatorname{Im} \sum_{j=1}^{n} P_{m(j)}(x, \xi, q)\, \overline{P}_m^{(j)}(x, \xi, q) > 0 \tag{4}$$

is satisfied. Further, wherever $x \in \overline{\Omega}$ *and there exists such* $0 \neq \xi \in R_n$ *that* $P_m(x, \xi, 0) = 0$, *the condition*

$$\lim_{q \to 0} \operatorname{Im} \frac{q}{|q|} \sum_{j=1}^{n} \left(P_{m(j)}^{(q)}(x, \xi, 0)\, P_m^{(j)}(x, \xi, 0) - P_{m(j)}(x, \xi, 0)\, P_m^{(j)(q)}(x, \xi, 0) \right) > 0 \tag{5}$$

is satisfied.

Here and below

$$P_m^{(q)}(x,\xi,0) = \frac{d}{dq} P_m(x,\xi,q) \quad \text{at } q=0.$$

Then there exist a constant K, independent of the function u, and a positive number d_0 such that estimate (2) is satisfied.

Proof. By carrying out a finite partition of unity in $\overline{\Omega}$, one can show that estimate (2) is local in character; we also note that, by choosing the number d_0 sufficiently large, it is easy to obtain that (2) does not depend on adding operators of order less than m. Consider $x = 0 \in \overline{\Omega}$ and $U_\delta = \{x;\ x\in\Omega,\ |x| < \delta\}$. Integrating by parts for $u \in C_0^\infty(U_\delta)$, one can obtain that

$$\int |P_m(x,D,q)u|^2\,dx \geq \int ([P^*,P]u)\,\overline{u}\,dx. \tag{6}$$

Here $P^* = P_m(x,D,\overline{q}) + P_{m-1}$, where P_{m-1} is an operator of order not greater than $m-1$, and $[P^*,P] = P^*P - PP^*$. Put $G_q(x,D)u = [P^*,P]u$; the principal part of the symbol of the operator $G_q(x,D)$ is determined by the formula

$$G_q(x,\xi) = 2\,\mathrm{Im}\sum_{j=1}^{n} P_{m(j)}(x,\xi,q)\,\overline{P}_m^{(j)}(x,\xi,q), \qquad \xi \in R_n. \tag{7}$$

Represent

$$G_q(x,\xi) = \sum_{j=0}^{2m-1} q^j G^{(j)}(x,\xi);$$

in view of the reality of the coefficients, it follows from (7) that $G^{(0)}(x,\xi) = 0$; therefore the expansion of G_q in powers of q has the form

$$\sum_{j=1}^{2m-1} q^j G^{(j)}(x,\xi),$$

where $G^{(j)}(x,\xi)$ are polynomials with continuous coefficients of order $2m-1-j$ in ξ. It can be shown, using conditions (4) and (5), that there exist constants C_1 and C_2 such that for all $0 \neq \xi \in R_n$ and $q \in \Theta$ the estimate

$$\left(|\xi|^2 + |q|^2\right)^{m-1} \leq C_1 |q|^{-1} G_q(0,\xi) + C_2 |P_m(0,\xi,q)|^2 \left(|\xi|^2 + |q|^2\right)^{-1}; \tag{8}$$

is satisfied. Multiplying (8) by $|\hat{u}(\xi)|^2$, where

$$\hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle} u(x)\,dx,$$

and integrating, we obtain

$$(2\pi)^{-n}\int |\hat{u}|^2\left(|\xi|^2+|q|^2\right)^{m-1}\,d\xi \le C_1|q|^{-1}\int (G_q(0,D)u)\,\overline{u}\,dx + C_2\|P_m(0,D,q)u\|_q^2,$$

where by $\|P_m(0,D,q)u\|_q^2$ we denote the expression

$$\int |P_m(0,\xi,q)u|^2(|\xi|^2+|q|^2)^{-1}\,d\xi.$$

Integrating by parts, using the Cauchy–Bunyakovsky inequality and taking (7) into account, one can show that, if δ is sufficiently small, then the estimate

$$\frac{1}{2}|q|(2\pi)^{-n}\int |\hat{u}|^2(|\xi|^2+|q|^2)^{m-1}\,d\xi \le (C_1+2C_2|q|^{-1})\int |P'_m(x,D,q)u|^2\,dx,$$

holds, whence (2) follows. The proof is complete.

To derive the estimates (2) for operators with complex-valued coefficients, we introduce an additional assumption.

Definition. We shall say that an operator P of the form (1) is **normal in the principal part** in Ω if its coefficients belong to $C^2(\overline{\Omega})$ and there exists a differential operator $Q(x,D,0)$ of order $m-1$ with coefficients in $C^1(\overline{\Omega})$ such that

$$C_{2m-1}(x,\xi,0) = 2\,\mathrm{Im}\sum_{j=1}^{n} P_{m(j)}(x,\xi,0)\overline{P}_m^{(j)}(x,\xi,0) =$$

$$= 2\,\mathrm{Re}\,P_m(x,\xi,0)\overline{Q}(x,\xi,0). \qquad (9)$$

Theorem 3. *Let P be an operator of the form (1), normal in the principal part. Suppose that at those points $x\in\Omega$ where there exist $0\ne\xi\in R_n$ and $0\ne q\in\Theta$ such that $P_m(x,\xi,q)=0$, condition (4) is satisfied. Further, where $x\in\overline{\Omega}$ and $P_m(x,\xi,0)=0$, $0\ne\xi\in R_n$, the condition*

$$\mathrm{Im}\left\{\left(\lim_{q\to 0}\frac{q}{|q|}\right)\sum_{j=1}^{n}\left(P_{m(j)}^{(q)}(x,\xi,0)\overline{P}_m^{(j)}(x,\xi,0) - P_m^{(j)(q)}(x,\xi,0)\overline{P}_{m(j)}(x,\xi,0)\right)\right\} -$$

$$-2\,\mathrm{Re}\left\{\lim_{q\to 0}\frac{q}{|q|}\left(P_m^{(q)}(x,\xi,0)\overline{Q}(x,\xi,0)\right)\right\} > 0. \qquad (10)$$

is satisfied. Then there exists a constant K such that the estimate (2) holds for functions in $C_0^\infty(\Omega)$.

The proof of Theorem 3 is, on the whole, analogous to the proof of Theorem 2.

Remark 1. If the coefficients of $P_m(x, D, q)$ are real and $\operatorname{Im} q \neq 0$, then for Q one may take an arbitrary operator with purely imaginary coefficients. In this case we can always satisfy inequality (10) by choosing $Q(x, \xi, 0) = -iMP_m^{(q)}(x, \xi, 0)$, where M is a positive or negative constant, depending on the sign of $\lim\limits_{q\to 0} \operatorname{Im} \dfrac{q}{|q|}$.

Remark 2. By virtue of the results of [2], p. 249, conditions (4) and (10) are invariant under changes of variables. Therefore all considerations of this note remain valid also when Ω is a precompact open set on a manifold.

Example. Let

$$P = -x\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 + iq\,\partial/\partial x_1 - q^2, \qquad x = (x_1, x_2)$$

with $q = i\tau$, $\tau \geq 0$, and let Ω be some neighborhood of the origin. Then

$$P(x, \xi, q) = x\xi_1^2 + \xi_2^2 - i\tau\xi_1 + \tau^2, \qquad \tau \geq 0.$$

At the point $x = 0$, $P(x, \xi, 0) = 0$ for $\xi_2 = 0$ and $\xi_1 \neq 0$. Note that at this same point $P^{(1)} = 0$ and $P^{(2)} = 0$ for $\xi_2 = 0$, $\xi_1 \neq 0$, and $q = 0$; therefore the operator P is not an operator of principal type in the sense of the definitions of [2]. Let us verify condition (5):

$$\left(\lim_{\tau\to 0} \operatorname{Im} \frac{i\tau}{\tau}\right)\left(-P^{(1)(q)}P_{(1)}\right) = \xi_1^2 > 0.$$

Next, $P(x, \xi, q) \neq 0$ for $q \neq 0$; consequently, for the operator P in Ω there exists an estimate of type (2). Using the averaging technique (see (2), § 2.4), we can pass from estimates in the space L_2 to estimates in the space $\mathcal{H}_{(s)}$.

Theorem 4. Let P be an operator of the form (1) with coefficients of the principal part in $C^\infty(\Omega)$, normal in the principal part. Suppose that conditions (4) and (10) of Theorem 3 are satisfied.

Then for any pair of real numbers s and t there exists a positive number d_0 such that if, for at least one q such that $|q| \geq d_0$, one has $P(x, D, q)u(x, q) = f(x, q) \in \mathcal{H}_{(s)}$, $u(x, q) \in \mathcal{H}_{(t)}$, then in fact $u \in \mathcal{H}_{(s+m-1)}$ for each such q, and for any such s and t and compact $K \subset \Omega$ there exists a positive constant $C_{s,t,k}$, independent of the function u and of the parameter $q \in \Theta$, such that the estimate holds

$$\|u\|_{(s+m-1)} \leq C_{s,t,k}\|P(x, D, q)u\|_{(s)}, \quad \text{if } \operatorname{supp} u \subset K.$$

Theorem 5. Let the assumptions of Theorem 4 be satisfied, and let Ω' be an open set $\Subset \Omega$. Suppose that $f(x, q) \in \mathcal{H}_{(s)}$ for all q such that $|q| \geq d_0$, for some positive number d_0. Denote by ${}^tP$ the operator adjoint to the operator P, i.e., such that

$$\int (P(x, D, q)u)v \, dx = \int u({}^tP(x, D, q)v) \, dx, \quad u, v \in C_0^\infty(\Omega).$$

Then one can find a function $v(x, q) \in \mathcal{H}_{(s+m-1)}$ for which ${}^tP(x, D, q)v = f$ in Ω' for all such q.

Theorem 6. Let the conditions of Theorem 4 be satisfied, and let Ω' be an open set $\Subset \Omega$. Suppose that the equation ${}^tP(x, D, q)v = 0$ has no nonzero solutions with support in Ω' for all q such that $|q| \geq d_0$, where d_0 is some positive number.

Then there exists a linear mapping $\mathcal{E}$ of the space $L_2(\Omega')$ into itself such that

$$P(x, D, q)\mathcal{E}f = f \text{ in } \Omega', \quad \text{if } f \in L_2(\Omega'),$$

$$\mathcal{E}P(x, D, q)u = u \text{ in } \Omega', \quad \text{if } u \in C_0^\infty(\Omega'),$$

$D^\alpha \mathcal{E}$ is a bounded operator in $L_2(\Omega')$, provided $|\alpha| < m$, for all such q.

The proofs of Theorems 5 and 6 can be obtained from the results of Theorem 4 by considering linear transformations in Hilbert spaces (see (2), § 8.8).

The author expresses his gratitude to V. V. Grushin for his attention and support.

Moscow State University
named after M. V. Lomonosov

Received
8 XI 1969

REFERENCES

1. I. S. Agranovich, M. I. Vishik, UMN, 19, no. 3 (117), 53 (1964).
2. L. Hörmander, *Linear Partial Differential Operators*, Moscow, 1965.
3. L. Hörmander, in: *Pseudodifferential Operators*, Moscow, 1967.
4. F. Treves, Acta Math., 101, 1 (1956).
5. C. Harvey, J. Math. and Mech., 16, no. 7 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.