LIMIT THEOREMS FOR SEMI-MARKOV PROCESSES WITH A COUNTABLE SET OF STATES

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Abstract

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MATHEMATICS

V. V. ANISIMOV

LIMIT THEOREMS FOR SEMI-MARKOV PROCESSES WITH A COUNTABLE SET OF STATES

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Let, for each $t \in (0, \infty)$, $\varkappa_t(s) \in \{1, 2, ...\}$ be a time-homogeneous right-continuous semi-Markov process (s.m.p.), which is specified, following ([1],[2]), by the matrix of transition probabilities

$$F_t(i, j, u) = P\{\varepsilon_{k+1} = j, \tau_t(\varepsilon_k) < u \mid \varepsilon_k = i\}, \quad i, j = 1, 2, \dots,$$

where $\varepsilon_k = \varkappa_t(\theta_t(k))$, $\tau_t(\varepsilon_k) = \theta_t(k+1) - \theta_t(k)$, $\theta_t(k)$ is the time of the k-th jump, i.e. $\theta_t(0) = 0$, and $\theta_t(k) = \min\{s : s > \theta_t(k-1), \varkappa_t(s) \neq \varepsilon_{k-1}\}$, $k \geq 1$.

Introduce the following random variables:

$$\beta_t(i, k) = \min\{n : n > \beta_t(i, k-1), \varepsilon_n = i\}, \quad k \geq 1$$

$$(\beta_t(i, 0) = 0),$$

and $\nu_t, \nu_t(i), \Omega_t(i), i = 1, 2, ...$, respectively, the total number of jumps of the s.m.p., the number of visits to state i, and the total time spent in i during time t.

Let, for each $t \in (0, \infty)$,

$$f_t^{(k)}(i, x), \quad x \in (0, \infty), \quad i = 1, 2, \dots, \quad k = 0, 1, 2, \dots,$$

be a family of mutually independent random variables, independent of the s.m.p. $\varkappa_t(s)$ (here and in what follows the quantities $\gamma^{(k)}$, $k = 0, 1, 2, ...$, denote random variables independent and identically distributed with γ). Introduce an additive functional $S(t)$ of the form

$$S(t)=\sum_{k=0}^{\nu_t-1} f_t(\varepsilon_k),$$

where $f_t(\varepsilon_k)=f_t^{(k)}(\varepsilon_k,\tau_t(\varepsilon_k))$, $k=0,1,2,....$

In what follows we shall assume that the embedded Markov chain, which, as is known, is specified by the matrix $P(t)=\|p_t(i,j)\|$, $i,j=1,2,...$, where $p_t(i,j)=F_t(i,j,\infty)$, $i,j=1,2,...$, has one positive class with stationary distribution $q_t(i)$, $i=1,2,...$, and

$$M\tau_t(i)=m_t(i)<\infty, \quad i=1,2,\dots,$$

$$A_t=\sum_{i=1}^{\infty} q_t(i)m_t(i)<\infty,$$

and the matrix $\bar{P}=\lim_{t\to\infty} P(t)$ also corresponds to a chain with one positive class.

Put

$$a_t=A_t^{-1}\sum_{i=1}^{\infty} q_t(i)m_t(f_i),$$

if $Mf_t(i,\tau_t(i))=m_t(f_i)<\infty$, $i=1,2,\dots$, and the given series converges absolutely, and $a_t=0$ otherwise. Denote

$$X_t(k)=\sum_{i=\beta_t(1,k)}^{\beta_t(1,k+1)-1} \tau_t(\varepsilon_i), \qquad \varphi_t(k)=\sum_{i=\beta_t(1,k)}^{\beta_t(1,k+1)-1} f_t(\varepsilon_i),$$

$$Y_t(k)=\varphi_t(k)-a_{tX}t(k), \qquad k=1,2,\dots$$

Theorem 1. If there exist $\gamma(t)$ and $b(t)$ such that

$$P\left\{\frac{1}{\gamma(t)}\sum_{k=1}^{V_t(1)}\left(X_t(k)-\frac{A_t}{q_t(1)}\right)<z\right\}\to F(z),* \tag{1}$$

$$P\left\{\frac{1}{b(t)}\sum_{k=1}^{V_t(1)} Y_t(k)<z\right\}\to \Phi(z), \tag{2}$$

$$\gamma(t)/t \to 0, \qquad A_t/\gamma(t)q_t(1) \to 0, \tag{3}$$

and for any $\varepsilon > 0,\ i = 1, 2, \dots$

$$P\{f_t(i, \tau_t(i)) > \varepsilon b(t)\} \to 0,$$

$$P\{a_t \tau_t(i) > \varepsilon b(t)\} \to 0$$

as $t \to \infty$, and $F(z)$ is a proper distribution function, then, independently of the initial state of the s.m.p.,

$$P\left\{\frac{1}{b(t)}(S(t) - ta_t) < z\right\} \to \Phi(z).$$

Corollary. If $a_t \neq 0$ and for any $k = 1, 2, \dots$, as $\alpha_t = o(1)$,

$$M \exp\{i\lambda\alpha_t(m_t(f_k))^{-1} f_t(k, \tau_t(k))\} = 1 + \alpha_t i\lambda(1 + o_k(1)),$$

then, under assumptions (1) and (3), the quantity $(ta_t)^{-1}S(t)$ tends in probability to one as $t \to \infty$, i.e., the law of large numbers is valid for $S(t)$.

Let us also note that under assumptions (1) and (3)

$$P\left\{\frac{A_t}{\gamma(t)q_t(1)}\left(v_t(1) - \frac{tq_t(1)}{A_t}\right) < z\right\} \to 1 - F(-z).$$

The significance of Theorem 1 is that it makes it possible to reduce the study of additive sums of an s.m.p. of the type $S(t)$ to the study of sums of a nonrandom number of independent identically distributed summands.

We apply the results obtained to the study of the vector of numbers of visits and sojourn times in the states of a certain finite subset $I = \{1, 2, \dots, r\}$ of the states of the s.m.p. To this end set

$$f_t(i, x) = \varphi_i + f_{ix},\ i = 1, \dots, r, \qquad f_t(i, x) = 0,\ i > r,$$

where $\varphi_i, f_i,\ i = 1, \dots, r$, are arbitrary real numbers.

Denote $\theta_t^*(0) = 0$, and

$$\theta_t^*(k) = \min\{\theta_t(l) : \theta_t(l) > \theta_t^*(k-1),\ \varepsilon_l \in I\}, \qquad k \geq 1.$$

Set $\tau_t^*(\varepsilon_k^*) = \theta_t^*(k+1) - \theta_t^*(k)$, where $\varepsilon_k^* = \chi_t(\theta_t^*(k))$. It is clear that

$$\tau_t^*(i) = \tau_t(i) + \tilde{\tau}_t(i), \qquad i \in I,$$

where

$$\tilde{\tau}_t(i) = \min\{s : \chi_t(s) \in I \mid \chi_t(0-0) = i,\ \chi_t(0) \neq i\}.$$

* Here the notation $V_t(j)$ denotes the integer part of $\dfrac{t}{A_t} q_t(j),\ j = 1, 2, \ldots$

Suppose that for each $j \in I$ there exist $b_t(j)$ and $B_t(j)$ such that the joint distribution

$$\left(\frac{1}{b_t(j)} \sum_{k=1}^{\nu_t(j)} \left(\tau_t^{(k)}(j) - m_t(j) \right),\ \frac{1}{B_t(j)} \sum_{k=1}^{\nu_t(j)} \left(\tilde{\tau}_t^{(k)}(j) - \widetilde{m}_t(j) \right) \right) \stackrel{\text{sl}}{\Longrightarrow} (\xi_j, \tilde{\xi}_j)^*, \quad (4)$$

$$D_t/t \to 0, \qquad A_t/D_t \to 0,$$

where

$$D_t = \max\{\sqrt{tA_t},\ b_t(j),\ B_t(j),\ j \in I\},$$

$$m_t = \max\{m_t(j),\ j \in I\}, \qquad \widetilde{m}_t(j) = M\tilde{\tau}(j).$$

Let

$$\psi_j(\lambda_1, \lambda_2) = M \exp\{i(\lambda_1 \xi_j + \lambda_2 \tilde{\xi}_j)\}.$$

Theorem 2. If condition (4) is fulfilled and $tm_t(A_t b_t)^{-1} \to 0$, then the joint distribution of the random vector

$$\left\{ \frac{m_t}{B_t} \left(\nu_t(1) - \frac{tq_t(1)}{A_t} \right),\ \sqrt{\frac{A_t}{t}} \left(\nu_t(1) - \frac{q_t(1)}{q_t(i)} \nu_t(i) \right), \right.$$

$$\left. i = 2, \ldots, r,\ g_t(k) \left(\Omega_t(k) - \frac{tq_t(k) m_t(k)}{A_t} \right),\ k = 1, \ldots, r \right\},$$

independently of the initial state, converges weakly to a distribution with characteristic function of the form

$$\exp\left\{-\frac{\sigma^2(\varphi_2,\dots,\varphi_r)}{2}\right\}\prod_{j=1}^{r}\psi_j(\rho_j f_j-\tilde{\rho}_j a(f),\,\alpha_j a(f)),$$

where the parameters $\varphi_i(f_i)$, $i=1,\dots,r$, correspond to the first (last) r components of the vector, $\sigma^2(\varphi_2,\dots,\varphi_r)$ is a nondegenerate quadratic form in the variables φ_i, $i=2,\dots,r$. Here

$$B_t=\max\left\{b_t(i),\,\frac{m_t}{A_t}B_t(i),\,i=1,\dots,r\right\},\qquad a(f)=\varphi_1+\sum_{i=1}^{r}c_i f_i,$$

$g_t(k)$, $k=1,\dots,r$, are normalizing factors, and $\rho_j,\tilde{\rho}_j,\alpha_j,c_i$, $j=1,\dots,r$, are certain constants.

The results obtained, in particular, as applied to countable Markov chains having an ergodic distribution, agree with the results of V. A. Volkonskii ([3]), and for a p.m.p. for which

$$F_t(i,j,u)=p(i,j)F(i,u),\qquad i,j=1,2,\dots,$$

under our assumptions they make it possible to obtain the corresponding results of H. Kesten ([4]).

Kiev State University
named after T. G. Shevchenko

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* In the sense of weak convergence of distribution functions.

Note: Figure translations are in progress. See original paper for figures.