PARAMETRIC REPRESENTATION OF UNIVALENT FUNCTIONS

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Abstract

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PARAMETRIC REPRESENTATION OF UNIVALENT FUNCTIONS

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Let S be the class of all holomorphic functions $w = f(z)$ univalent in the disk $E = \{z: |z| < 1\}$, normalized by the conditions $f(0) = 0,\ f'(0) = 1$. In the present paper we give a solution of the problem of the parametric representation of the class S, establishing necessary and sufficient conditions for a function $w = f(z)$ to belong to the class S. We then indicate one application of the result obtained to the solution of extremal problems in the theory of univalent functions and note the connection between extremal problems in the class S and the class P of all functions $w = h(z)$ holomorphic in the disk E, $h(0) = 1$, with positive real part.

1. Let $\mathfrak{M}$ be the class of all nondecreasing functions $\mu(x, y)$ of two variables in the domain $x \geq 0,\ -\pi \leq y \leq \pi$, normalized by the conditions $\mu(x, -\pi) = \mu(0, y) = 0,\ \mu(x, \pi) = x$.

It follows immediately from the definition of the class $\mathfrak{M}$ that for each fixed $y,\ -\pi \leq y \leq \pi$, the functions $\mu(x, y)$ are absolutely continuous with respect to x, and consequently, for almost all $x,\ x > 0$, there exists the derivative $\mu'_x(x, y)$, which is a measurable function of the variable x for each fixed value of y, and a nondecreasing function of the variable $y,\ -\pi \leq y \leq \pi$, for fixed $x,\ x > 0$, normalized by the condition $\mu'_x(x, -\pi) = 0,\ \mu'_x(x, \pi) = 1$.

We shall say that a sequence $\mu_n(x, y)$ $(n = 1, 2, ...)$ of functions of the class $\mathfrak{M}$ converges to a function $\mu(x, y) \in \mathfrak{M}$ if at all points of continuity of the function $\mu(x, y)$ one has $\lim\limits_{n\to\infty} \mu_n = \mu$.

The class $\mathfrak{M}$ is compact in itself with respect to the convergence of sequences of functions from $\mathfrak{M}$ defined above.

Denote by Φ the set of all continuous functions $f(z, x, y)$ in the domain $E \times [0, \infty) \times [-\pi, \pi]$, analytic with respect to z in the disk E and satisfying the condition $|f(z, x, y)| \leq e^{-x} K(r)$, where $K(r)$ is a constant depending only on $r = |z| < 1$.

Let $f(z, x, y)$ be an arbitrary function of the class Φ, and let μ_n be an arbitrary sequence of functions of the class $\mathfrak{M}$ converging to a function $\mu(x,y) \in \mathfrak{M}$. Then:

1) there exists, uniformly with respect to x, $0 \le x \le A$, and $z \in E_r = \{z : |z| \le r < 1\}$, the limit

$$\lim_{n\to\infty} \int_0^x \int_{-\pi}^{\pi} f(z,x,y)\, d\mu_n(x,y) = \int_0^x \int_{-\pi}^{\pi} f(z,x,y)\, d\mu(x,y);$$

2) the Stieltjes integrals

$$\int_0^\infty \int_{-\pi}^{\pi} f(z,x,y)\, d\mu(x,y), \qquad \mu \in \mathfrak{M},$$

converge uniformly inside E, uniformly with respect to the class $\mathfrak{M}$.

From this there follows directly the existence, uniformly inside E, of the limit

$$\lim_{n\to\infty} \int_0^\infty \int_{-\pi}^{\pi} f(z,x,y)\, d\mu_n(x,y) = \int_0^\infty \int_{-\pi}^{\pi} f(z,x,y)\, d\mu(x,y).$$

2. Consider the differential equation

$$\frac{dw}{dx} = -w \int_{-\pi}^{\pi} g(w,y)\, d\mu'_x(x,y), \tag{1}$$

where $g(w,y) = (1 + e^{iy}w)/(1 - e^{iy}w)$, with the initial condition

$$w(x)\big|_{x=0} = z, \qquad z \in E.$$

Here the function $\mu(x,y) \in \mathfrak{M}$, and the integral in (1) is understood in the Stieltjes sense.

We shall denote by $f(z,x;\mu)$ the solution of the differential equation (1) satisfying the initial condition.

Theorem 1. *In order that the function $w = f(z)$ belong to the class S, it is necessary and sufficient that it can be represented in the form*

$$f(z) = \lim_{x\to\infty} e^x f(z,x;\mu), \qquad \mu \in \mathfrak{M}. \tag{2}$$

We outline the proof of Theorem 1. Let $\mu(x,y)$ be an arbitrary function of the class $\mathfrak{M}$. Replace equation (1) with the initial condition by the integral equation

$$w = z \exp\left\{-\int_0^x \int_{-\pi}^{\pi} g(w,y)\, d\mu(x,y)\right\}, \tag{3}$$

which is obtained from (1) by division by w and integration with respect to x from 0 to x. Solving (3) by the method of successive approximations (cf., for example, ([1]), pp. 96-97), we find that the solution $w = f(z, x; \mu)$ of equation (3) is regular in the disk E and continuous for $0 < x < \infty$, and, moreover,

$$f(0, x; \mu) = 0, \qquad f'_z(0, x; \mu) = e^{-x}.$$

By virtue of an easily proved uniqueness theorem for the solution of equation (1), it follows that the function $f(z, x; \mu)$ is univalent in E for each fixed value of x in $[0, \infty)$. It remains to establish the existence, uniformly in z inside E, of the limit (2). For this purpose we substitute $f(z, x; \mu)$ into equation (1) and rewrite it in the form

$$\left[e^x f(z, x; \mu)\right]'_x = e^x f(z, x; \mu)\,[1 - g(f(z, x; \mu), y)], \tag{4}$$

noting at the same time that the function standing on the right-hand side of equation (4) belongs to the class Φ*.

Integrating (4) with respect to x from 0 to x and passing to the limit as x tends to infinity, we arrive at the conclusion that the function $f(z)$ obtained from formula (2) belongs to the class S.

Now let $f(z)$ be an arbitrary function of the class S. We shall show that it can be obtained by formula (2) with a suitably chosen function $\mu(x, y)$ from the class $\mathfrak{M}$. To this end denote by $\mathfrak{M}'$ the subclass of the class $\mathfrak{M}$ consisting of functions $\mu(x, y)$ such that

$$\int_{-\pi}^{\pi} g(w, y)\, d\mu'_x(x, y) = g(w, y(x)).$$

By Loewner' s theorem ([2]) (see also ([1]), p. 95), the totality of functions $f(z)$ obtained by formula (2), when $\mu(x, y)$ runs through the class $\mathfrak{M}'$, forms a subclass S' of the class S, everywhere dense in S with respect to uniform convergence inside the disk E.

* This follows from the estimates $|f(z, x; \mu)| \le |z|, \quad |f(z, x; \mu)| \le \dfrac{e^{-x}|z|}{(1 - |z|)^2}$.

Choose a sequence $f_n(z)$ of functions of the class S', converging uniformly inside E to the function $f(z)$. To the sequence $f_n(z)$ there corresponds a sequence $\mu_n(x, y)$ of functions of the class $\mathfrak{M}$ such that

$$f_n(z) = \lim_{x\to\infty} e^x f(z, x; \mu_n).$$

From $\mu_n(x, y)$ one can choose a subsequence converging, in the sense indicated earlier, to some function $\mu^*(x, y)$ of the class $\mathfrak{M}$. Now, using the propositions of Section 1, it is not difficult to show that the function $f(z)$ itself can be obtained by formula (2) for $\mu = \mu^*$.

From the Riesz-Herglotz theorem ([3]) it follows that the function

$$h(w, x) = \int_{-\pi}^{\pi} g(w, y)\, d\mu_x'(x, y), \qquad \mu \in \mathfrak{M}, \tag{5}$$

for each fixed x, $0 < x < \infty$, is regular in w in the disk $|w| < 1$ and has there a positive real part. Consequently, from the known differential equation of K. Löwner-P. P. Kufarev ([4]) and relation (2), all functions of the class S can be obtained.

3. From the identity

$$dw/dx = -wh(w, x), \qquad w = f(z, x; \mu), \tag{6}$$

where $h(w, x)$ is computed by formula (5), taking (2) into account, there immediately follow the relations in the class S that are needed for the subsequent arguments:

$$f(z) = z \exp\left\{\int_0^{|z|} \frac{1 - F(w, \rho)}{\operatorname{Re} F(w, \rho)} \frac{d\rho}{\rho}\right\}, \tag{7}$$

$$f'(z) = \exp\left\{\int_0^{|z|} \frac{1 - F(w, \rho) - wF_w'(w, \rho)}{\operatorname{Re} F(w, \rho)} \frac{d\rho}{\rho}\right\}. \tag{8}$$

Here $F(w, \rho) = h(f(z, x(\rho); \mu), x(\rho))$, $\rho = |f(z, x; \mu)|^*$.

Theorem 2. Let z_0 be a fixed point of the disk E, and let $\alpha, \beta, \gamma, \delta$ be arbitrary real numbers. Then, for the functional

$$I(f) = \alpha \ln\left|\frac{f(z_0)}{z_0}\right| + \beta \arg\frac{f(z_0)}{z_0} + \gamma \ln|f'(z_0)| + \delta \arg f'(z_0), \tag{9}$$

defined on the class S, the following sharp estimates hold:

$$\int_0^{|z_0|} \varphi(\xi^-, \eta^-) \frac{d\rho}{\rho} \le I(f) \le \int_0^{|z_0|} \varphi(\xi^+, \eta^+) \frac{d\rho}{\rho}, \tag{10}$$

where $(\xi^\pm, \eta^\pm)$ are the points of the circle

$$\xi^2 - 2a(\rho)\xi + \eta^2 + 1 = 0, \qquad a = (1 + \rho^2)(1 - \rho^2)^{-1},$$

at which the function

$$\varphi(\xi, \eta) = a - \alpha - \gamma + (\alpha + \gamma)/\xi - \gamma\xi - \eta(\delta + (\beta + \delta)/\xi) \tag{11}$$

attains its maximum (minimum) value.

Equality in (10) is realized, for example, for functions $f(z)$ of the class S having the form

$$f(z) = \lim_{x\to\infty} e^x f(z,x),$$

where $w = f(z,x)$ is the solution of the equation

$$w'_x = -wg(w, y^{\pm}(x)), \qquad w(0) = z,$$

where

$$y^{\pm}(x) = \arcsin \eta^{\pm}[\xi^{\pm}(a^2-1)^{1/2}]^{-1} + \int_0^x \eta^{\pm}\, dx - \arg z_0,$$

and $\rho = \rho(x)$ is determined from the relation

$$(\ln\rho)'_x = -\xi^{\pm}, \qquad \rho(0) = |z_0|.$$

* From (6) it follows that $\rho(x)$ is a monotonically decreasing function, since $(\ln\rho)'_x = -\operatorname{Re} h < 0$.

From formulas (7), (8) it follows that the problem of estimating the functional $I(f)$ of the form (9) on the class S is equivalent to finding the extremum of the real functional $J(h) = \Psi(h(z), zh'(z))/\operatorname{Re} h(z)$, $z = \rho e^{i\varphi} \in E$ and fixed, where $\Psi(\omega, w) = (\alpha+\gamma)(1-\operatorname{Re}\omega) - (\beta+\delta)\operatorname{Im}\omega - \gamma\operatorname{Re} w - \delta\operatorname{Im} w$, on the class P (see, on this question, works [5,6]) and to the subsequent integration of the result with respect to ρ from 0 to $|z_0|$. A similar connection between the extremal problems of the classes S and P also holds for other problems in the theory of functions of a complex variable.

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Note: Figure translations are in progress. See original paper for figures.