ON INTEGRAL OPERATORS

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Abstract

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1°. Let E be a real B-space, and let $S(a,b)$ be the space of measurable functions. In 1938 L. V. Kantorovich and B. Z. Vulikh ([1]) introduced and studied a class of operators acting from E into $S(a,b)$ and having the form

$$(T(x))(s) = (x, \varphi(s)), \quad x \in E, \tag{1}$$

where $\varphi(s)$ is a function* with values in the conjugate space E^*.

For separable or reflexive B-spaces E, L. V. Kantorovich and B. Z. Vulikh showed that an operator $T : E \to S(a,b)$ has the form (1) if and only if T is a *bo*-linear operator ([2]).

Operators of the form (1) are a generalization of integral operators: if E and E^* are spaces of measurable functions and $(x, x^*) = \int x(t)x^*(t)\,dt$, then (1) has the form

$$(Tx)(s) = \int K(s,t)x(t)\,dt, \quad x \in E, \tag{2}$$

where the kernel $K(s,t) = \varphi(s)$ is measurable in t for almost all $s \in (a,b)$ and satisfies the condition $K(s,\cdot) \in E^*,\ s \in (a,b)$.

It is known ([1–3]) that, under additional restrictions on E and on the operator T, the kernel in (2) is jointly measurable in the variables; in ([1,2]) it was shown that if the operator T is *bo*-linear from $L_p(a,b)$ into $L_q(a,b)$, then the kernel in (2) is jointly measurable in the variables and, moreover, satisfies the condition $\|\|K(s,t)\|_{p'}\|_q < \infty,\ 1/p + 1/p' = 1$.

In ([3]) this result was generalized to the case of *bo*-linear operators from $X(0,1)$ into $Y(0,1)$, where $X(0,1), Y(0,1)$ are spaces of measurable functions belonging respectively to the classes P and $P \cup Q$, introduced by D. A. Vladimirov ([3]).

It should be noted that the proofs of the propositions cited from ([1–3]) rely essentially on the separability of the space from which the operator acts.

The difficulties that arise in representing the function $\varphi(s)$ in the form of a jointly measurable kernel, and the restrictions on spaces and operators associated with them, are due to the fact that the function $\varphi(s)$ in (1) is weakly measurable. These difficulties largely disappear if the function $\varphi(s)$ is strongly measurable ([4],[5]).

In this connection there arises the problem of conditions for representability of an operator in the form (1) with a strongly measurable function $\varphi(s)$. Below a solution of this problem is given for operators acting from an arbitrary normed space (real or complex) into the space $S(X,\mu)$, where (X,μ) is an arbitrary space with a σ-finite measure. Further, a broad class of spaces $F(Y,\nu)$ of ν-measurable functions is indicated ((Y,ν) is an arbitrary space with a σ-finite measure) possessing the property that strongly measurable functions $\varphi(s)$ with values in these spaces generate, by the equality $\varphi(s)=\overline{K(s,\cdot)}$, kernels measurable jointly in the variables, and propositions are given on

* From (1) it follows that the function $\varphi(s)$ is weakly measurable.

integral representability of linear operators acting from normed spaces of measurable functions into spaces of measurable functions.

2°. Let E be a normed space (real or complex), L a linear manifold in E, and (X,Σ,μ) a space with a σ-finite measure. A linear operator $T: L\to S(X,\mu)$ will be called a C-operator if

$$(Tx)(s)=(x,\varphi(s)),\quad x\in L,$$

where $\varphi(s): X\to E^*$ is a strongly μ-measurable function, which we shall call the kernel of the operator T. In what follows we shall assume that the closure of L has a topological complement in E ([6]).

We shall call a set $H\subset L^\infty(X,\mu)$ σ-weakly compact if there exists an at most countable set of disjoint sets X_n, $n\in I$, such that $\mu(X_n)<\infty$, $\mu(X\setminus\bigcup_{n\in I}X_n)=0$, and the sets $P_n(H)$, $n\in I$, are weakly compact in $L^\infty(X,\mu)$, where

$$P_{nf}=\chi_{X_n}f,$$

χ_{X_n} is the characteristic function of the set X_n.

A linear bounded operator $Q: L\to L^\infty(X,\mu)$ will be called σ-weakly completely continuous if, for every bounded set $G\subset L$, the set $Q(G)$ is σ-weakly compact in $L^\infty(X,\mu)$.

Theorem 1. *A linear operator $T: L\to S(X,\mu)$ is a C-operator if and only if there exists a function $\Lambda\in S(X,\mu)$, $\Lambda>0$, such that the operator*

$$(\tau x)(s)=(Tx)(s)/\Lambda(s),\quad x\in L,$$

is a σ-weakly completely continuous operator from L into $L^\infty(X,\mu)$.

Remark. If L is everywhere dense in E, then the kernel of the operator T is determined uniquely up to μ-equivalence.

Corollary. *Let $E = H$, where H is a Hilbert space. An operator $T : H \to L_2(X, \mu)$ is a Hilbert-Schmidt operator* ([7]) *if and only if the operator T has a majorant* ([8]) *belonging to $L_2(X, \mu)$.*

In the case when H and $L_2(X, \mu)$ are separable, the corollary coincides with Proposition 2.10 ([9]).

Theorem 2. *Let E be reflexive, and let $T : L \to S(X, \mu)$ be a linear operator. The following assertions are equivalent: 1) T has an abstract norm* ([2])*; 2) T has a majorant* ([8])*; 3) T is a C-operator; 4) T has the form*

$$(Tx)(s) = (x, \varphi(s)),$$

where φ is weakly μ-measurable; T is a b_0-linear operator, i.e., it maps null sequences from L into sequences converging to zero μ-almost everywhere.

3°. Let (Y, Ξ, ν) be a space with a σ-finite measure, and let $F(Y, \nu)$ be a normed space of ν-measurable functions*. We shall say that $F(Y, \nu)$ is σ-embedded in $L_1(Y, \nu)$ if there exists an at most countable set of disjoint sets Y_n, $n \in I$, such that $\mu(Y_n) < \infty$,

$$\mu\Big(Y \setminus \bigcup_{n \in I} Y_n\Big) = 0$$

and

$$\|P_{nf}\|_{L_1(Y,\nu)} \leq c_n \|f\|_{F(Y,\nu)},$$

where $P_{nf} = \chi_{Y_n} f$ and c_n does not depend on f.

Lemma. *Let $\varphi(s) : X \to F(Y, \nu)$ be a strongly μ-measurable function and let $F(Y, \nu)$ be σ-embedded in $L_1(Y, \nu)$. Then there exists a $(\mu \times \nu)$-measurable function $K(s, t)$ such that $\varphi(s) = K(s, \cdot)$.*

Theorem 3. *Let $G(Y, \nu)$, $[G(Y, \nu)]^*$ be spaces of ν-measurable functions and*

$$(x, x^*) = \int_Y x(t)\overline{x^*(t)}\, d\nu(t), \quad x \in G(Y, \nu),\ x^* \in [G(Y, \nu)]^*,$$

$[G(Y, \nu)]^*$ is σ-embedded in $L_1(Y, \nu)$, and L is a linear manifold in $G(Y, \nu)$. If the operator $T : L \to S(X, \mu)$ is a C-operator, then the operator T is an integral operator with a $(\mu \times \nu)$-measurable kernel $K(s, t)$, satisfying the condition

$$\|K(s, \cdot)\|_{[G(Y,\nu)]^*} \in S(X, \mu).$$

* By a space $Z(Y, \nu)$ of measurable functions, here and below in the article we mean a space of classes f of ν-equivalent functions satisfying the conditions: a) if $f \in Z(Y, \nu)$, then also $|f| \in Z(Y, \nu)$, and $\|f\|_Z = \||f|\|_Z$; b) if $f \in Z$ and $|g| \leq |f|$, $g \in S(Y, \nu)$, then $g \in Z$ and $\|g\|_Z \leq \|f\|_Z$.

4°. **Theorem 4.** Let 1) $H(Y, \nu)$, $G(Y, \nu)$, $F(X, \mu)$ be normed spaces of measurable functions; 2) $L = H(Y, \nu) \cap G(Y, \nu)$ be everywhere dense in $H(Y, \nu)$ and $G(Y, \nu)$; 3) $G(Y, \nu)$ satisfy the conditions of the preceding Theorem 3 and be reflexive; 4) $H(Y,\nu)$ be a Banach space; 5) $T : H(Y,\nu) \to F(X,\mu)$ be a regular operator. In order that the operator T be an integral operator with a $(\mu \times \nu)$-measurable kernel satisfying the condition $\|K(s,\cdot)\|_{[G(Y,\nu)]^*} \in S(X,\mu)$, it is necessary and sufficient that there exist a function $\Lambda \in S(X,\mu)$ such that, for all $f \in L$, $|(Tf)(s)| \le \Lambda(s)\|f\|_{G(Y,\nu)}$ for μ-almost all $s \in X$.

5°. The results obtained in (8,9) for operators in separable spaces can be extended to the nonseparable case. Thus, assertions 1), 3) of Theorems 1-5 (8) remain valid also in the case when μ, μ_0 are not separable $*$ σ-finite measures. Assertions 2) of Theorems 1-5 (8) are formulated taking into account the separability of the range of operators of type (SC). We give, for example, analogues of assertions 2) of Theorems 1, 2 (8).

Theorem 5. Let the measure μ be σ-finite and not purely atomic. An operator T is an operator of type (SC) if and only if the adjoint operator T^* is densely defined, the residual spectrum of the operator T^* contains 0, and the range of the closure of the operator T is separable.

Theorem 6. Let the measure μ be σ-finite and not purely atomic. An operator T is an operator of type (C) if and only if the adjoint operator T^* is densely defined and there exists a symmetric operator A such that $A \subseteq T^*$, the residual spectrum of the operator A contains 0, and the range of the adjoint operator A^* is separable.

6°. **Theorem 7.** An operator $T : L_2(Y,\nu) \to L_2(X,\mu)$ is an integral operator with a $(\mu \times \nu)$-measurable Hilbert–Schmidt kernel if and only if there exists a majorant (8) of the operator T belonging to $L_2(X,\mu)$.

In the case $X = Y = (a,b)$, $\mu = \nu$ is Lebesgue measure, the theorem follows from a theorem of L. V. Kantorovich and B. Z. Vulikh ((2), p. 332; see also (3), p. 773).

Corollary 1. Let $A : L_2(Z,\xi) \to L_2(Y,\nu)$ and $B : L_2(X,\mu) \to L_2(W,\eta)$ be linear bounded operators, and let $T : L_2(Y,\nu) \to L_2(X,\mu)$ be an integral operator with a $(\mu \times \nu)$-measurable Hilbert–Schmidt kernel. Then BTA is an integral operator with an $(\eta \times \xi)$-measurable Hilbert–Schmidt kernel.

Corollary 2. An operator $T : L_2(Y,\nu) \to L_2(X,\mu)$ is a Hilbert–Schmidt operator (7) if and only if the operator T is an integral operator with a $(\mu \times \nu)$-measurable Hilbert–Schmidt kernel.

In the separable case the assertion of the corollary is well known ((4), p. 102).

7°. The results of the present article can be extended to the case where X is a separable locally compact space and μ is a Radon measure.

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* That is, $L_2(X, \mu)$, $L_2(X_0, \mu_0)$ are not separable.

Note: Figure translations are in progress. See original paper for figures.