LIMIT THEOREMS FOR FUNCTIONALS OF THE PROCESS OF STEP SUMS OF RANDOM VARIABLES DEFINED ON A SEMI-MARKOV PROCESS WITH A FINITE SET OF STATES

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Abstract

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LIMIT THEOREMS FOR FUNCTIONALS OF THE PROCESS OF STEP SUMS OF RANDOM VARIABLES DEFINED ON A SEMI-MARKOV PROCESS WITH A FINITE SET OF STATES

(Presented by Academician A. N. Kolmogorov on 11 V 1970)

Let $T_j,\ j = 1,2$, be independent collections of random variables, defined as follows: $T_1 = \{\eta_n,\ n = 0,1,2,\ldots\}$ is a homogeneous Markov chain with finite set of states $H = \{1,2,\ldots,m\}$ and transition-probability matrix $\|p_{ij}\|_{i,j=1}$; $T_2 = \{(\tau(n,i),\gamma(n,i)),\ n \geq 0,\ i \in H\}$ is a collection of independent random vectors, taking values in $[0,\infty) \times (-\infty,\infty)$, whose distributions do not depend on n.

The random process

$$\eta(t) = \eta_{\nu(t)}, \qquad t \geq 0,$$

where

$$\nu(t) = \max\left(n : \sum_{k=1}^{n} \tau(k-1,\eta_{k-1}) \leq t\right),$$

is called a semi-Markov process ([1]).

We require that $T_j,\ j = 1,2$, satisfy the following regularity condition:

(A_1) : 1) T_1 is ergodic (we denote its stationary distribution by $q_j,\ j = 1,2,\ldots,m$);

$$2)\ \sum_{i=1}^{m} P\{\tau(0,i) > 0\} > 0.$$

For each $t > 0$, let us introduce the random process

$$\xi_t(s) = \sum_{k=1}^{\nu(st)} \gamma(k-1, \eta_{k-1}), \qquad s \in [0,1],$$

which it is natural to call the process of step sums of random variables defined on the semi-Markov process $\eta(t),\ t \geq 0$.

Let $D_{[0,1]}$ be the space of functions on $[0,1]$ without discontinuities of the second kind, right-continuous with the uniform metric

$$\rho(x(s), y(s)) = \sup_{s\in[0,1]} |x(s) - y(s)|,$$

$$\mu(C) = P\{w(s) \in C\}, \qquad C \in \mathfrak{B};$$

here $\mathfrak{B}$ is the σ-algebra of Borel sets in $D_{[0,1]}$, and $w(s),\ s \in [0,1]$, is a Wiener process continuous with probability 1.

It is obvious from the construction that for all $t > 0$, with probability 1 the trajectories of the random process $\xi_t(s),\ s \in [0,1]$, belong to $D_{[0,1]}$.

Definition. We shall say that a measurable functional $f(\cdot)$, defined on $D_{[0,1]}$, is continuous in the uniform topology if there exists a set $C \in \mathfrak{B}$ such that $\mu(C) = 1$ and for all $x_n(s) \in D_{[0,1]}$,

$n \geqslant 0$, if

$$x_0(s) \in C,\ \rho(x_n(s), x_0(s)) \to 0 \quad \text{as } n \to \infty,$$

then

$$\lim_{n\to\infty} f(x_n(s)) = f(x_0(s)).$$

A number of examples of μ-continuous functionals in the uniform topology are given in ([2]).

Theorem 1. *If condition (A_1) and $(A_2)'$ are satisfied: $M|\gamma(0,i)|^2 < \infty$, $M\tau(0,i)^2 < \infty$, $i \in H$, then all finite-dimensional distributions of the random process*

$$w_t(s) = t^{-1/2}\left(\xi_t(s) - \frac{b}{a}st\right), \qquad s \in [0,1],$$

as $t \to \infty$, converge weakly (at continuity points) to the corresponding finite-dimensional distributions of the random process

$$w_0(s) = \sigma w(s), \qquad s \in [0,1];$$

here

$$a = \sum_{i=1}^{m} q_i M\tau(0,i), \qquad b = \sum_{i=1}^{m} q_i M\gamma(0,i),$$

$$\sigma^2 = \lim_{n\to\infty} (an)^{-1} D \sum_{k=1}^{n} \left(\gamma(k-1,\eta_{k-1}) - \frac{b}{a}\tau(k-1,\eta_{k-1}) \right).$$

Remark 1. In ([3]) a simple method is given for finding the constant through $\|p_{ij}\|_{i,j=1}^m$ and $M\gamma(0,i)^{k'}\tau(0,i)^{k''}$, $i \in H$, $k', k'' \geqslant 0$, $k'+k'' \leqslant 2$, reducing to the solution of a finite system of linear equations.

Theorem 2. *If the conditions* (A_j), $j = 1,2$, *are satisfied, then for all functionals* $f(\cdot)$ *on* $D_{[0,1]}$ *that are* μ*-continuous in the uniform topology,*

$$P\{f(w_t(s)) < u\} \to P\{f(w_0(s)) < u\} \quad \text{as } t \to \infty$$

for all continuity points of the distribution function standing on the right.

Remark 2. For the case when condition (B) is satisfied:
1) $\tau(0,i) = 1$, $i \in H$ with probability 1;
2) $m = 1$ (control by the Markov chain is absent).
The corresponding results are contained, for example, in ([2]).

The proof of Theorem 1 is carried out analogously to how this is done for one-dimensional distributions in ([4]).

The proof of Theorem 2 contains two stages.

It is not difficult to prove that, when the conditions (A_j), $j = 1,2$, are satisfied,

$$\sup_{s\in[0,1]} \left| \frac{\nu(st)}{t} - a^{-1}s \right| \xrightarrow{P} 0 \quad \text{as } t \to \infty.$$

Next, using a simple assertion,

Lemma 1. *If, for a sequence of random processes* $\xi_n(s)$, $s \geqslant 0$, $n = 0,1,\ldots$, *whose trajectories with probability 1 belong to the space* $D_{[0,\infty)}$ *of functions on* $[0,\infty)$ *without discontinuities of the second kind and continuous from the right, and a sequence of random processes* $\nu_n(s)$, $s \geqslant 0$, $n = 0,1,\ldots$, *taking nonnegative values with probability 1 and whose trajectories with probability 1 belong to* $D_{[0,\infty)}$, *the relations*

$$\text{a)}\quad \rho_n(t) = \sup_{s\in[0,t]} |\xi_n(s) - \xi_0(s)| \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

where $\xi_0(s),\ s \geq 0$, is a random process continuous with probability 1, $t \geq 0$;

b)

$$\hat{\rho}_n(T) = \sup_{s\in[0,T]} |v_n(s) - v_0(s)| \xrightarrow{P} 0 \quad \text{as } n \to \infty,\ T \geq 0;$$

c)

$$P\left\{\sup_{s\in[0,T]} v_0(s) \geq t\right\} \to 0 \quad \text{as } t \to \infty,$$

then

$$\sup_{s\in[0,T]} |\xi_n(v_n(s)) - \xi_0(v_0(s))| \xrightarrow{P} 0 \quad \text{as } n \to \infty;$$

and by the identity

$$\{v(t) \geq x\} = \left\{\sum_{k=1}^{[x]} \tau(k-1, \eta_{k-1}) \leq t\right\},$$

the proof can be reduced to the case where all $\tau(0,i) = 1,\ i \in H$, with probability 1.

The remainder of the proof is carried out analogously to that given in [2] for the case in which condition (B) holds.

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Note: Figure translations are in progress. See original paper for figures.