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Abstract

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF ONE CLASS OF INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY AND MATHEMATICAL PHYSICS

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The solution of mixed problems of the theory of elasticity ([1], [2]) and of mathematical physics for a layer with a circular line of change of boundary conditions can be reduced to the study of the integral equation

$$K_s q \equiv \int_0^a k_s(r,\rho)\, q(\rho)\, \rho\, d\rho = f(r), \qquad 0 \le r \le a, \tag{1}$$

with kernel

$$k_s(r,\rho) = \int_0^\infty uK(u) J_s(ur) J_s(u\rho)\, du; \tag{2}$$

$J_s(z)$ is the Bessel function of integer index s.

Some dual integral equations ([3]) are also reduced to equation (1).

The problem of hydrodynamic impact considered in ([4]) is equivalent to the equation

$$\Delta K_0 q = f(r), \qquad 0 \le r \le a.$$

Δ is the axisymmetric Laplace operator.

Equation (1) is a spatial analogue of the convolution equation on a finite interval ([5]). If a certain plane mixed problem of mathematical physics gives rise to a convolution equation on an interval, then the same mixed problem in the spatial formulation with a step condition of boundary conditions on a circle of radius a leads to equation (1).

In papers ([1–3]) a method was proposed for the asymptotic study of equation (1) as $a \to 0$. In the same papers an expansion of the solution in the parameter a is given. In ([3]) another method of solving this equation was proposed, based on the study of dual integral equations. However, both methods prove ineffective if $a \to \infty$. In ([6]) a method was proposed for constructing the zero term of the asymptotics of equation (1) as $a \to \infty$. In the author' s paper ([5]), under the assumption of meromorphy of the function $K(u)$, a regular representation of the solution of the indicated equation for large a is constructed.

In the present note a theorem is given that answers a number of general questions concerning the properties of the solution of equation (1) for large a. An analytic form of the solution is given and the domain of its existence and uniqueness is indicated.

1. Let E denote the set of smooth contours Γ—the boundaries of all possible convex neighborhoods $V(\Gamma)$ of the point $-i\infty$, lying entirely in the lower half-plane and having no common points with the real axis. By $\mu(\Gamma)$ we shall denote the distance from $V(\Gamma)$ to the real axis. We shall write $\Gamma_2 > \Gamma_1$ if $V(\Gamma_2) \supset V(\Gamma_1)$ and the distance between the contours Γ_2 and Γ_1 is bounded below by a fixed number. It is not difficult to see that for $\Gamma_1 \in E$ one can easily construct, and moreover not uniquely, a contour $\Gamma_2 \in E$

and such that $\Gamma_2 > \Gamma_1$. The contour Γ_2 can be constructed by deforming, for example, Γ_1 in such a way that its points are displaced along the outward normal by an amount not reaching the distance from Γ_1 to the real axis.

Let there be found a contour $\Gamma_1 \in E$ with equation $z = x + iy(x)$ $(y(x) < -\mu(\Gamma_1),\ |y(x)| = O(x^\varepsilon),\ x \to \infty,\ \varepsilon \geqslant 1)$ such that the function $K(z)$ is regular in the domain $\Omega:\ |\operatorname{Im} z| \leqslant -y(x),\ |x| < \infty$, and is continuous on Γ_1.

In addition, we shall assume that $K(z)$ is an even function, real on the real axis, and possessing the asymptotic behavior

$$K(z) = c^2 z^{-2\gamma}[1 + O(z^{-\alpha})], \qquad z \in \Omega + \Gamma_1, \quad |z| \to \infty,$$
$$0 < \gamma < 1, \qquad \alpha > 0. \tag{3}$$

In this case, for $K(z)$ the representation

$$K(z) = K_-(z) K_+(z) \tag{4}$$

holds.

$K_-(z)$ is regular in the domain $\Omega \cup \operatorname{Im} z < 0$, $K_+(z)$, respectively, in $\Omega \cup \operatorname{Im} z > 0$, and, moreover, the estimate

$$K_+(z), \quad K_-(z) \sim c z^{-\gamma}, \qquad z \in \Omega, \quad |z| \to \infty \tag{5}$$

is valid.

Denote by A the set of functions $\varphi(z)$, regular in the domain $S = \Omega \cap \operatorname{Im} z \leqslant -\delta$ ($\delta > 0$ is an arbitrarily small fixed number) and admitting the representation (5)

$$\varphi(z) = \psi(z) z^{-1}, \qquad \max_{z\in S} |\psi(z)| < \infty, \qquad 0 < \delta < \mu(\Gamma_1).$$

If in A we introduce a norm by the relation

$$\|\varphi\|_A = \max_{z\in S} |\psi(z)|, \tag{6}$$

then A becomes a Banach space.

On elements of A we define the operator

$$F(a,z) = \frac{1}{(2\pi i)^2} \int_{\Gamma_2} \int_{\Gamma_1} \frac{p(t_2,t_1)\varphi(t_1)\,dt_1 dt_2}{z - t_2}, \qquad z \in \Gamma_3; \tag{7}$$

$$p(t_2,t_1) = K_+(t_2)\,[R_1(t_1) + R_2(t_2)]/(t_2^2 - t_1^2)\,K_+(t_1),$$

$$\Gamma_3 > \Gamma_2 > \Gamma_1, \qquad \Gamma_k \in E, \qquad k = 1,2,3,$$

$$R_1(t) = tI_{s+1}(ita)I_s^{-1}(ita) - t, \qquad R_2(t) = tK_{s+1}(ita)K_s^{-1}(ita) - t; \tag{8}$$

$I_s(t), K_s(t)$ are modified Bessel functions of order s.

It is not difficult to establish that the operator $F(a,z)$ acts continuously in A. The norm of the operator A, acting from the contour Γ_1 to the contour $\Gamma_3 > \Gamma_2 > \Gamma_1$, is estimated by the inequality

$$\|F\|_{\Gamma_1\to\Gamma_3} \leqslant \max_{z\in\Gamma_3} \frac{1}{4\pi^2} \int_{\Gamma_2} \int_{\Gamma_1} \left| \frac{zp(t_2,t_1)}{(z-t_2)t_1} \right| |dt_1 dt_2|. \tag{9}$$

The same operator, acting in A from the contour Γ_3 to Γ_1, we represent in the form

$$F(a,z)\varphi = \frac{1}{(2\pi i)^2} \int_{\Gamma_4} \int_{\Gamma_3} \frac{p(t_4,t_3)\varphi(t_3)\,dt_3 dt_4}{z - t_4} - \frac{1}{2\pi i} \int_{\Gamma_3} p(z,t_3)\varphi(t_3)\,dt_3 +$$

$$+\varphi(z)\frac{R_1(z) + R_2(z)}{2z}, \qquad z \in \Gamma_1, \quad \Gamma_3 < \Gamma_4 \in E, \tag{10}$$

where

$$\|F\|_{\Gamma_3\to\Gamma_1} \leqslant \max_{z\in\Gamma_1}\left\{\frac{1}{4\pi^2}\int_{\Gamma_4}\int_{\Gamma_3}\left|\frac{zp(t_4,t_3)}{(z-t_4)t_2}\right||dt_3 dt_4|+\right.$$

$$\left.+\frac{1}{2\pi}\int_{\Gamma_3}\left|\frac{zp(z,t_3)}{t_3}\right||dt_3|+\left|\frac{R_1(z)+R_2(z)}{2z}\right|\right\}. \tag{11}$$

The functions under the integral sign in relations (7), (10) are, obviously, analytic in S; therefore the contours of integration may be deformed. In doing so, the form of the operator will not change if the contour does not intersect a singularity of the integrand. We complete E by piecewise-smooth Γ' s. The integration in (7), (10) must be performed in the order in which the differentials occur, i.e., first the inner integral is evaluated, then the outer one.

Theorem. The unique solution in $L_p(0,a)$ $(p>1)$ of the integral equation (1) with right-hand side $f(r)\in c_2^{\lambda}(0,a)$ $(\lambda\ge\gamma)$ for values $a>a_0$ is given by the relation

$$q(r)=\int_0^\infty \frac{\Phi(\eta)\eta J_s(\eta r)\,d\eta}{K(\eta)}+\sum_{n=0}^{\infty}(-1)^n S(r)F^{nD}; \tag{12}$$

a_0 is the greatest root of the equation

$$1=\inf_{\gamma_k\in E}\|F\|_{\gamma_3\to\gamma_1}\|F\|_{\gamma_1\to\gamma_3}. \tag{13}$$

Here the infimum is taken over all contours $\gamma_k\in E$ such that $\gamma_1<\gamma_2<\gamma_3<\gamma_4$. Moreover, the relation

$$q(r)(a-r)^{\gamma}\in c(0,a), \tag{14}$$

holds, and for the partial sum

$$q_m(r)=\int_0^\infty \frac{\Phi(\eta)\eta J_s(\eta r)\,d\eta}{K(\eta)}+\sum_{n=0}^{m}(-1)^n S(r)F^{nD} \tag{15}$$

the asymptotic estimate

$$\left[q(r)-q_m(r)\right](a-r)^{\gamma}=O\left[a^{-2(m+1)}\right],\qquad a\to\infty. \tag{16}$$

holds.

Here the following notation has been introduced:

$$S(r)f = \frac{1}{2\pi i} \int_{-\infty - i\varepsilon_1}^{\infty - i\varepsilon_1} \frac{I_s(itr) f(t)\, dt}{I_s(ita) K_+(t)}, \qquad 0 < \varepsilon_1 < \mu(\Gamma_1),$$

$$\psi(\tau) = \int_0^\infty \left[\frac{\tau K_{s+1}(i\tau a) J_s(\eta a)}{K_s(i\tau a)} + i\eta J_{s+1}(\eta a) \right] \frac{\eta \Phi(\eta)\, d\eta}{(\eta^2 - \tau^2) K(\eta)}, \qquad \operatorname{Im} \tau < 0, \tag{17}$$

$$D(t) = \frac{1}{2\pi i} \int_\Gamma \frac{K_+(\tau)\psi(\tau)\, d\tau}{t - \tau}, \qquad \Gamma_1 \le \Gamma \in E, \quad t \in \Gamma_2 > \Gamma,$$

$$f(r) = \int_0^\infty \eta \Phi(\eta) J_s(\eta r)\, d\eta,$$

$F^n(a, z)$ is the n-th iteration of the operator $F(a, z)$; $c_2^\lambda(0, a)$ is the set of functions whose second derivative satisfies a Hölder condition with exponent λ on $[0, a]$; $c(0, a)$ is the set of continuous functions on $[0, a]$.

2. As an example, consider equation (1) for $s = 0$ in the case ([6]) when

$$K(z) = (z^2 + b^2)^{-0.5}, \qquad K_+(z) = (b - iz)^{-0.5}, \qquad K_-(z) = (b + iz)^{-0.5},$$

and the cut connects the infinitely remote point along the imaginary axis with the points $+ib$, $-ib$, ib, respectively, for each function. The branches of the functions are chosen from the condition

$$K_+(z) \to z^{-0.5} \exp(i\pi/4), \qquad K_-(z) \to z^{-0.5} \exp(-i\pi/4), \qquad z \to \infty.$$

Let us compute a_0. As the contours γ_k we take the contours described by the equations

$$\gamma_k = x - i\left(|x| + B_k\right), \qquad k = 1, 2, 3, 4, \qquad b > B_1 > B_2 > B_3 > B_4 > 0.$$

For $B_4 \ge 1$ we obtain a substantially simplified, and therefore overestimated, bound of the form

$$\|F\|_{\Gamma_3 \to \Gamma_1} \|F\|_{\Gamma_1 \to \Gamma_3} < Q(B_1, B_2, B_3, b) \left[Q(B_3, B_4, B_1, b) + \frac{108}{\pi}(B_1 - B_2)\sqrt{B_1 B_3} + 14 B_3^{-2} \right] a^{-4}.$$

Here

$$Q(B_1, B_2, B_3, b) = \frac{18}{\pi^2}\left[\frac{2\pi}{\sqrt{B_1}} + \frac{3\sqrt{B_1 + b}}{B_2}\ln(B_1 + B_2)\right] \times \left[\frac{2}{B_1 - B_2} + \pi\left(B_2 - B_3 + B_1^{-0.5}\right)\right].$$

For $b = 5$, setting $B_k = 5 - k$ ($k = 1, 2, 3, 4$) and carrying out the calculations, we obtain $a_0 < 11.3$. Thus, in the present case, the series (12) represents a solution of equation (1) in the interval

$$11.3 \leq a < \infty.$$

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Note: Figure translations are in progress. See original paper for figures.