THE CENTER AND SPREAD OF A PROBABILITY DISTRIBUTION

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Abstract

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MATHEMATICS

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THE CENTER AND SPREAD OF A PROBABILITY DISTRIBUTION

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I. The success of constructing a general theory of limit theorems for sums of independent random variables, in which the traditional condition of uniform infinitesimality of the summands is not used (see ([1])), is to a considerable extent due to the use of new characteristics of distributions, called centers. Since characteristics of this type, which are a generalization of the concepts of mathematical expectation and variance, may prove useful also outside the theory of limit theorems, it seems useful to acquaint specialists in the field of probability theory with them.

II. In what follows the following notation is used: $F = F_\xi$, $f = f_\xi$—respectively the distribution function and the characteristic function of the random variable ξ; $Q(x, \xi)$—the concentration function of the distribution F_ξ;

$$\Delta(\xi) = \sup\{r : \min_{|t|\le r} |f(t)| > 0\} \qquad (0 < \Delta(\xi) \le \infty).$$

Definition. Let ξ be some random variable and let $r > 0$ be a number such that the function $\log f(t)$ is defined and integrable on the interval $|t| \le r$. The quantities

$$C^r(\xi) = \frac{2}{r^2}\int_0^r \operatorname{Im}\log f(t)\,dt, \qquad B^r(\xi) = -\frac{6}{r^2}\int_0^r \operatorname{Re}\log f(t)\,dt$$

will be called respectively the r-center and the r-spread* (briefly, the center and spread) of the random variable ξ.

III. Many of the properties of centers and spreads presented in this section are obvious in character and are given only for completeness of the argument in favor of the naturalness of the proposed definition. These characteristics are very convenient for the effective choice of normalizing constants in linear transformations of random variables and therefore can be used successfully in limit theorems. Centers, moreover, successfully combine the additive properties of mathematical expectations (true, only for independent summands) and the centering properties of medians. Incidentally, all this will not be difficult to notice when becoming acquainted with the properties of centers and spreads set out below.

1°. The quantities $C^r(\xi), B^r(\xi)$ exist for every $r < \Delta(\xi)$.

2°. If $N > 0$ is a number such that $q = Q(N, \xi) > 1/2$, then

$$\Delta(\xi) \geq \sup\{x : \min_{|t| \leq x} \operatorname{Re} f_\xi(t) > 0\} \geq \sqrt{4q-2}/N. \tag{1}$$

3°. If the mathematical expectation $\mathbf{E}\xi$ and the variance $\mathbf{D}\xi$ exist (finite or infinite), then as $r \to 0$

$$C^r(\xi) \to \mathbf{E}\xi, \qquad B^r(\xi) \to \mathbf{D}\xi. \tag{2}$$

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* A similar functional was used for somewhat different purposes by A. Ya. Khinchin.

4°. The following equalities hold:

$$\Delta(-\xi) = \Delta(\xi); \qquad C^r(-\xi) = -C^r(\xi), \qquad B^r(-\xi) = B^r(\xi)$$

for all $r < \Delta(\xi)$.

If λ is some positive number, then

$$\Delta(\lambda\xi) = \Delta(\xi)/\lambda$$

and for all $r < \Delta(\lambda\xi)$

$$C^r(\lambda\xi) = \lambda C^{r\lambda}(\xi), \qquad B^r(\lambda\xi) = \lambda^2 B^{r\lambda}(\xi).$$

If ξ has the normal (γ, σ^2)-distribution, then $\Delta(\xi) = \infty$ and for all r

$$C^r(\xi) = \gamma, \qquad B^r(\xi) = \sigma^2.$$

5°. Let $\xi = \xi_1 + \xi_2 + ...$ be a series of independent random variables, converging with probability 1. Then

$$\Delta(\xi) = \inf_j \Delta(\xi_j); \tag{3}$$

for any $r < \Delta(\xi)$ the centers and spreads of the summands ξ_j exist, and moreover

$$C^r(\xi) = \sum_j C^r(\xi_j), \qquad B^r(\xi) = \sum_j B^r(\xi_j). \tag{4}$$

6°. Let the sequence $\xi_n \xrightarrow{p} \xi$ (in probability) as $n \to \infty$. Then

$$\Delta(\xi_n) \to \Delta(\xi); \tag{5}$$

for each $r < \Delta(\xi)$ and all sufficiently large n the centers and spreads of the random variables ξ_n exist, and as $n \to \infty$

$$C^r(\xi_n) \to C^r(\xi), \qquad B^r(\xi_n) \to B^r(\xi). \tag{6}$$

7°. If the sequence ξ_n is such that for some random variable ξ conditions (5) and (6) are fulfilled, then the distributions of the random variables ξ_n form a compact set with respect to weak convergence.

If, in addition, $\Delta(\xi) = \infty$, then $\xi_n \xrightarrow{p} \xi$ as $n \to \infty$.

8°. For a sufficiently high degree of concentration of the distribution of the random variable ξ, the center ξ turns out to be close both to the median $\mu(\xi)$ and to the truncated mean of this random variable, while the spread is close to the truncated variance. Namely, let $r < \Delta(\xi)$, $\varepsilon > 0$, $q = Q(\varepsilon, \xi) \geqslant \alpha > 1/2$, and let

$$A = \{|\xi| < \varepsilon\}, \qquad a_\varepsilon(\xi) = \mathbf{E}(\xi I_A), \qquad \sigma^2_\varepsilon(\xi) = \mathbf{D}(\xi I_A),$$

where I_A is the indicator of the event A. Then

$$|C^r(\xi) - \mu(\xi)| < K_1(\varepsilon r + 1 - q)/r,$$

$$|C^r(\xi) - a_\varepsilon(\xi)| < K_2(\varepsilon^3 r^3 + 1 - q)/r,$$

$$|B^r(\xi) - \sigma^2_\varepsilon(\xi)| < K_3(\varepsilon^4 r^4 + 1 - q)/r^2,$$

where the constants K_i depend only on i, α.

IV. We shall illustrate the possibilities for using centers and spreads by means of several theorems.

Theorem 1. For all $r < \Delta(\xi)$ the inequalities

$$0.08\,\mathbf{E}s\Big(\frac{r}{11.2}\,\bar{\xi}\Big) \leqslant r^2 B^r(\xi) \leqslant -3\log\left\{1 - 4\mathbf{E}s\left(\frac{r}{\sqrt{2}}\,\bar{\xi}\right)\right\},$$

hold, where $s(x) = \min(x^2, 1)$ and $\bar{\xi} = \xi - C^r(\xi)$. The upper estimate of $r^2 B^r(\xi)$ will be,

naturally, meaningful only for those values of r for which the right-hand side of the inequality is finite.

Corollary. For every $r < \Delta(\xi)$ and any $X > 0$ the inequality holds

$$P\{|\xi - C^r(\xi)| > X\} < 1970\,B^r(\xi)/\min(X^2, r^{-2}).$$

Taking into account property 3° of the preceding item, it is not difficult to note that this inequality is a generalization of the well-known Chebyshev inequality.

Theorem 2. In order that the series of independent random variables $\xi_1 + \xi_2 + \ldots$ converge (converge absolutely) with probability 1, it is necessary and sufficient that the following conditions be fulfilled:

a) $\Delta = \inf_j \Delta(\xi_j) > 0$;

b) for some $s < \Delta$ the series converge

$$\sum_j C^s(\xi_j) \quad \left(\text{respectively } \sum_j |C^s(\xi_j)|\right), \qquad \sum_j B^s(\xi_j).$$

Corollary. Any series of independent random variables that converges with probability 1 can be transformed into an absolutely convergent one if each summand is centered by its own s-center.

Theorem 3. The set of distribution functions $\mathfrak{S} = \{F_\xi\}$ will be compact with respect to weak convergence if and only if:

a) $\Delta = \inf_{\mathfrak{S}} \Delta(\xi) > 0$;

b) there exist a sequence $r_1, r_2, \ldots, r_k < \Delta$, $r_k \to 0$, and constants $a_k > 0$, $b_k > 0$, with $b_k \to 0$ as $k \to \infty$, such that

$$|C^{r_k}(\xi)| < a_k, \qquad r_k^2 B^{r_k}(\xi) < b_k \quad \text{for all } F_\xi \in \mathfrak{S}.$$

V. Let us note that the definition of center and spread can be varied, while preserving either all or the greater part of the properties indicated above. Namely, one may consider as centers and spreads quantities of the form

$$C_\sigma^r(\xi) = \frac{\displaystyle\int_0^r \operatorname{Im}\log f_\xi(t)\, d\sigma(t)}{\displaystyle\int_0^r t\, d\sigma(t)},$$

$$B_\sigma^r(\xi) = -\,2\,\frac{\displaystyle\int_0^r \operatorname{Re}\log f_\xi(t)\, d\sigma(t)}{\displaystyle\int_0^r t^2\, d\sigma(t)},$$

where σ is such a measure on $(0, \infty)$ for which the integral appearing in the denominator of the definition of C_σ^r is finite and positive (σ may depend on r).

Centers and spreads defined with the aid of measures σ that have a density differ little from one another in their properties. The simplest form of centers and spreads is obtained if for σ one chooses the degenerate distribution with a jump at the point r. In normalizing random variables, this variant of centers and spreads can be used no less successfully than the variant with $\sigma \equiv t$.

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1. V. M. Zolotarev, C. R., **270**, 899 (1970).

Note: Figure translations are in progress. See original paper for figures.