

NONABELIAN $\backslash(\text{CL}\backslash)$ -GROUPS

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Abstract

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MATHEMATICS

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NONABELIAN CL -GROUPS

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Abelian groups in which every servient subgroup is complemented were studied in the work of S. N. Chernikov ⁽¹⁾. In connection with that work, the question arose of the structure of abelian groups in which all maximal subgroups of some fixed rank are complemented. This problem was solved in papers by one of the authors ^(2,3). In them the notion of an L_r -closed subgroup of an abelian group was introduced, and abelian CL_r -groups were studied, i.e., abelian groups in which all L_r -closed subgroups are complemented. Let us give the definition of an L_r -closed subgroup of an abelian group ⁽³⁾. Here r may take values equal to an arbitrary natural number or ∞ ; moreover, for $r = \infty$ the notation $s(X) = r$ means that the rank $s(X)$ of the group X is infinite, while the notation $s(X) < r$ means that the rank of the group X is finite. By $\Omega_p(X)$ is meant the set of all elements of an arbitrary abelian group X whose order does not exceed the prime number p , and by $T(X)$ the maximal periodic subgroup of X .

In the case when $s(A) < r$, the L_r -closed subgroup B of the abelian group A always coincides with A : $B = A$. If A is a p -primary abelian group with $s(A) \geq r$, then a subgroup B is L_r -closed if and only if $s(B) = r$ and, for every subgroup C satisfying the relations $B \subseteq C$, $\Omega_p(B) = \Omega_p(C)$, the equality $B = C$ holds. A subgroup B of a periodic abelian group A is L_r -closed if all its Sylow p -subgroups are L_r -closed in the corresponding Sylow p -subgroups of the group A . Finally, if A is a mixed abelian group with $s(A) \geq r$, then a subgroup B is L_r -closed if and only if $s(B) = r$, the subgroup $T(B)$ is L_k -closed in the subgroup $T(A)$ for $k = s(T(B))$, and for every subgroup C satisfying the relations

$$B \subseteq C, \quad T(B) = T(C), \quad T(C/B) = C/B,$$

the equality $B = C$ holds. We note that in the case when r is a natural number and A is an arbitrary abelian group with $s(A) \geq r$, the L_r -closed subgroups of the group A are precisely all maximal subgroups of rank r of A ⁽³⁾.

A subgroup B of an abelian group A is called L -closed if B is an L_r -closed subgroup for some r (we recall that r may be any natural number or the symbol

∞); an abelian group is called a *CL*-group if every one of its *L*-closed subgroups is complemented. From the results of ^(2,3) it follows that a group *A* is an abelian *CL*-group if and only if *A* is a group of one of the following types:

a)

$$A = \sum_{\alpha \in M} \{a_\alpha\}, \quad p^{k-1}a_\alpha \neq 0, \quad p^{k+1}a_\alpha = 0 \quad (\alpha \in M).$$

where the numbers p, k do not depend on α , and the number p is prime;

- b) *A* is a periodic abelian group whose Sylow *P*-subgroups are either complete groups or groups of type a);
- c) $A = R + A_1 + A_2 + \dots + A_n$, where *R* is a complete abelian group, and A_1, A_2, \dots, A_n are pairwise isomorphic torsion-free groups of rank 1.

The aim of the present work is to carry over the noted results concerning abelian *CL*-groups to the case of nonabelian groups. In doing so we shall follow the path proposed by S. N. Chernikov in his paper ⁽⁴⁾, namely, we shall consider groups satisfying the following condition (we shall call it the *CL* condition).

Condition *CL*. Every *L*-closed subgroup of an arbitrary maximal abelian subgroup of the group is complemented in the group itself.

It is clear that the class of arbitrary groups satisfying condition *CL* includes, in particular, the class of abelian *CL*-groups.

It would be natural first to study the class of nilpotent groups with condition *CL*. However, in doing so we do not go beyond the class of abelian *CL*-groups, since a nilpotent group with condition *CL* is abelian—this follows from the results of the present article. Exactly the same is true for *ZA*-groups satisfying condition *CL*. A broader class than the class of *ZA*-groups is the well-known class of *CI**-groups, i.e. the class of groups possessing a completely ordered ascending invariant series with cyclic factors. The class of *CI**-groups with condition *CL* already turns out to be considerably wider than the class of abelian *CL*-groups (a consequence of Theorem 3). Moreover, in the course of studying the class of *CI**-groups with condition *CL* it became clear that the only property of *CI**-groups needed by us is the following property: a *CI**-group possesses an invariant maximal abelian subgroup (a consequence of Theorem 1). Therefore, in what follows we shall consider groups in which at least one maximal abelian subgroup is invariant. We formulate the definition that specifies the object of study of our work.

Definition. A group *G* is called a *CL*-group if it satisfies condition *CL* and possesses an invariant maximal abelian subgroup.

Theorem 1 and its corollary show that the class of groups possessing an invariant maximal abelian subgroup is sufficiently broad.

Theorem 1. *If a group *G* is an extension of an abelian group *A* by a *CI**-group, then every maximal invariant abelian subgroup *B* of the group *G* containing the*

subgroup A is a maximal abelian subgroup of the group G .

Corollary. *The class of groups possessing an invariant maximal abelian subgroup contains all metabelian (solvable of length two) groups and all CI^* -groups (in particular, ZA -groups).*

Let us note that Theorem 1 is analogous to Lemma 1 of (5). Theorems 2 and 3 give a complete description of nonabelian CL -groups.

Theorem 2. *A periodic group G is a CL -group if and only if it decomposes into a product of abelian subgroups A, B, C :*

$$G = A \times (B\lambda C),$$

where: 1) the Sylow p -subgroups of the group G are abelian CL -groups for every prime number p ; 2) the subgroup B contains no elements of order two and decomposes into a direct product of groups of rank 1 that are invariant in G ; 3) if a subgroup H of B is invariant in G , then its centralizer in the subgroup C is a direct summand in C .

Let us note that the groups A, B, C occurring in the formulation of Theorem 2 are abelian CL -groups. Indeed, according to item 1), for every prime number p the Sylow p -subgroup $G_p = A_p \times B_p \times C_p$ of the group G is an abelian CL -group, and therefore the Sylow p -subgroups A_p, B_p, C_p of the groups A, B, C are abelian CL -groups. Hence the groups A, B, C are also CL -groups.

Theorem 3. *A nonperiodic nonabelian group G is a CL -group if and only if it belongs to one of the following types of groups:*

- a) $G = P \times (N\lambda\{b\})$, where P is a periodic complete abelian group containing no elements of order two, N is a nonperiodic abelian CL -group containing no elements of order two, and $b^2 = 1, b^{-1}xb = x^{-1}$ for all $x \in N$;
- b) $G = P \times (R\lambda C)$, where P is a periodic complete abelian group, R is the direct product of subgroups R_α ($\alpha \in M$) isomorphic to the additive group of all rational numbers and invariant in the group G , and C is a free abelian group of finite rank, every element of which induces on each subgroup R_α ($\alpha \in M$) either the identity automorphism or an automorphism of infinite order.

Consequence. A ZA -group satisfying the condition CL is abelian. Nonabelian CI^* -groups satisfying the condition CL are exhausted by the groups indicated in Theorem 2 and in item a) of Theorem 3.

From the description of CL -groups given above it follows that the class of CL -groups is essentially broader than the class of completely factorizable groups (groups in which all subgroups are complemented). We note that a CL -group is completely factorizable if and only if it is periodic and the orders of its elements are not divisible by squares of prime numbers. This follows from Theorems 2, 3 and from the description of completely factorizable groups (6).

From Theorems 2, 3 one can also extract a number of properties of CL -groups analogous to the known properties of completely factorizable groups. In particular, every CL -group G decomposes into a semidirect product of two abelian groups, with the invariant factor being the direct product of groups of rank 1 invariant in the group G , and all Sylow p -subgroups of the group G being abelian. As is known ⁽⁶⁾, every completely factorizable group decomposes into a semidirect product of two abelian groups, with the invariant factor being the direct product of cyclic subgroups of prime orders invariant in it; the Sylow p -subgroups of a completely factorizable group are abelian (even elementary abelian). Further, every periodic CL -group is isomorphically embeddable in the complete direct product of groups of two types: 1) cyclic or quasicyclic 2-groups, 2) periodic subgroups of the holomorph of a cyclic p -group or a quasicyclic p -group for $p > 2$. For an analogous assertion for completely factorizable groups see ⁽⁶⁾.

Remarks. 1. The notion of an L -closed subgroup of an abelian group is close to the notion of a weakly servient subgroup ⁽⁷⁾. In particular, an L -closed subgroup of an abelian group is weakly servient (but not conversely), and for abelian groups with primary periodic part the two noted notions are equivalent. Abelian groups in which all weakly servient subgroups are complemented were studied in ⁽⁸⁾; it turned out that they coincide exactly with abelian CL -groups.

2. Since every servient subgroup of an abelian group is weakly servient, the abelian groups in which all weakly servient subgroups are complemented constitute a subclass of the class of abelian groups with complemented servient subgroups studied by S. N. Chernikov ⁽¹⁾. In ⁽¹⁾ a complete description of the structure of abelian groups of this kind was given. It also completely determines, in particular, the structure of abelian groups in which all weakly servient subgroups are complemented, obtained in ⁽⁸⁾ (without taking account of the results from ⁽¹⁾).

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