# DECREASING SEQUENCES OF MEASURABLE PARTITIONS AND THEIR APPLICATIONS

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**Abstract**

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*MATHEMATICS*

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# DECREASING SEQUENCES OF MEASURABLE PARTITIONS AND THEIR APPLICATIONS

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Decreasing sequences of measurable partitions arise in various situations, among which one should single out the theory of endomorphisms of spaces with measure, the theory of dynamical systems in which time is an inductive limit of compact groups, and, finally, the theory of nonmeasurable partitions, which has a number of important applications to dynamical systems and rings of operators.

The main purpose of this work is to construct a rather unexpected example of two nonisomorphic sequences that differ by a not very obvious limiting invariant. The example is described in § 4°; on its basis a Markov regular process is constructed that negatively solves Lévy' s problem (see ([1], [2])) for homogeneous conditional measures.

**1°. Decreasing sequences.** Let, on the interval $[0, 1]$ with Lebesgue measure (or on a Lebesgue space $X$ with continuous measure), there be given a monotonically decreasing sequence of measurable partitions $\xi = \{\xi_i\}$, $\xi_{i+1} < \xi_i$, $i = 1, \dots$ ([3]). The problem is posed of the metric classification of such sequences. Leaving aside the relatively uninteresting case of finite sequences, this problem can be made concrete as follows.

Let two decreasing sequences on $X$, $\xi = \{\xi_i\}_{i=1}^{\infty}$ and $\xi' = \{\xi'_i\}_{i=1}^{\infty}$, be finitely isomorphic, i.e., for every natural $n$ there exists an automorphism (= a one-to-one mod 0 measurable invertible measure-preserving transformation) $T_n$ of $X$ such that $T_n\xi_i = \xi'_i$, $i = 1, \dots, n$. When does there exist an automorphism $T$ for which $T\xi_i = \xi'_i$, $i = 1, \dots$? In other words, what are the limiting invariants of a decreasing sequence of measurable partitions? The problem is easily reformulated in terms of algebras of measurable sets or rings of operators.

The first limiting invariant is obvious—the measurable intersection

$$\bigwedge_{i=1}^{\infty} \xi_i.$$

Its exclusion (i.e., reduction of the problem to the case

$$\bigwedge_{i=1}^{\infty} \xi_i = \nu,$$

where $\nu$ is the trivial partition) is carried out similarly to the decomposition of an automorphism into ergodic components ([3]). New and useful in many questions is another invariant—the accompanying partition. Let $\xi = \{\xi_i\}$ be a finite or infinite sequence of measurable partitions. The **accompanying partition** $\Xi(\xi)$ is the finest of all measurable partitions that remain fixed under the action of the group of automorphisms with respect to which all $\xi_i$ are invariant (in other words, $\Xi(\xi)$ is the partition into the ergodic components of this group). It is clear that any automorphism $T$ carrying $\xi$ into $\xi'$ carries $\Xi(\xi)$ into $\Xi(\xi')$ in a fixed way, i.e., the induced isomorphism

$$T_{\Xi} : X/\Xi(\xi) \to X/\Xi(\xi')$$

does not depend on $T$.

If $\Xi(\xi) = \varepsilon$ (where $\varepsilon$ is the partition into individual points mod 0), then the sequence $\xi$ is called **absolutely nonhomogeneous** (example: a Bernoulli $S$-endomorphism with two states and probabilities $p \neq 1/2$ and $q$; then $\{S^{-i}\varepsilon\}_{i=1}^{\infty}$ is absolutely nonhomogeneous). The most difficult and interesting

in the classification problem, the case of a **homogeneous sequence** $\Xi(\xi) = \nu$. To it, using the accompanying partition, one can reduce the general classification problem. Finite segments of a homogeneous sequence look very simple.

**Lemma 1.** *Every finite homogeneous decreasing sequence of measurable partitions $\{\xi_i\}_{i=1}^{n}$ is determined, up to isomorphism, by $n$ numbers $r_i$, $i = 1, \dots, n$, taking natural values or $+\infty$: $r_i$ is the number of points in an element of the partition $\xi_i/\xi_{i-1}$, $i = 1, \dots, n$, $\xi_0 = \varepsilon$, which is the same for almost all elements ($X/\xi_i$ is assumed to be non-discrete, $i = 1, \dots, n$).*

A homogeneous sequence $\xi = \{\xi_i\}_{i=1}^{\infty}$ is called **dyadic** if $\xi_i \downarrow \nu$ and $r_i = 2$ for all $i$. Almost every element of the partition $\xi_i$ in such a sequence consists of $2^i$ points of equal (conditional) measure, $i = 1, \dots$. Dyadic sequences are, by the lemma, finitely isomorphic. The central difficulty of the general problem is their classification. The simplest example of a dyadic sequence is the **standard dyadic sequence**: $\xi^{(0)} = \{\xi_i^{(0)}\}_{i=1}^{\infty}$, $X = [0, 1]$; an element of $\xi_i^{(0)}$ is the set containing, together with a given number, all numbers differing from it only in the first $i$ digits of the binary expansion.

In ([4]) it was proved that any two dyadic sequences $\xi$ and $\xi'$ are lacunarily isomorphic, i.e., there exists a sequence of natural numbers $i_k \nearrow \infty$ such that $\{\xi_{i_k}\}_{k=1}^{\infty}$ and $\{\xi'_{i_k}\}_{k=1}^{\infty}$ are isomorphic. This result is very important for the trajectory theory of dynamical systems (see ([5])) and for a number of questions in the theory of rings of operators. However, the statement formulated in ([4]), p. 18, without proof and not used there, a more general assertion on the isomorphism of all dyadic sequences, turned out to be erroneous*. From it, of course, as noted in ([4]), lacunary isomorphism would also follow.

**Theorem 1.** *There exists a dyadic sequence not isomorphic to the standard dyadic sequence.*

An outline of the proof of this theorem is given in the following sections. The construction of such a sequence serves as the basis for examples. In fact, there exists a continuum of pairwise non-isomorphic dyadic sequences (see item 5°).

2°. **Universal projector.** Let $\eta$ be a measurable partition of a Lebesgue space $(X, \mu)$; $L(X; K)$ is the space of all measurable mappings of $X$ into the metric compactum $K$. A universal projector with respect to $\eta$ is a mapping $\mathcal{P}_\eta : L(X; K) \to L(X; S(K))$, where $S(K)$ is the simplex of all nonnegative normalized measures on $K$, and $\mathcal{P}_\eta$ is constructed as follows: let $f \in L(X; K)$; considering the restriction $f$ to almost every element $C$ of the partition $\eta$, we obtain a uniquely defined mod 0 measurable family $\{f_C\}_{C\in\eta}$, $f_C : C \to K$, and if $\{\mu_C\}$ is the canonical system of measures ($\mu_C$ is the conditional measure on $C$ –see ([3])), and $f\mu_C$ is the image of $\mu_C$ under the mapping $f_C$, then $(\mathcal{P}_\eta f)(x) = f\mu_{C(x)}$, where $C(x)$ is the element of $\eta$ containing $x$.

Let $\xi = \{\xi_i\}_{i=1}^n$ be a decreasing sequence of partitions. The universal projector for $\xi$ is defined as follows: $\mathcal{P}_\xi = \mathcal{P}_{\xi_n} \cdots \mathcal{P}_{\xi_i}$. $\mathcal{P}_\xi$ is an operator from $L(X; K)$ to $L(X; S_n(K))$, where $S_n(K) = S(S_{n-1}(K))$**. In terms of $\mathcal{P}_\xi$ one can easily describe invariant properties of sets and functions with respect to $\xi$, in particular, give a classification of finite decreasing sequences (a more traditional approach to such a classification is in ([8])). The usefulness of universal projectors is seen from the following theorem.

* Unfortunately, this formulation found its way into the works ([6]), p. 51, and ([7]), p. 277). The author offers deep apologies to the authors of ([6,7]).

** $S(K)$ is a metrizable compactum in the weak topology.

**Theorem 2.** In order that a dyadic sequence $\xi = \{\xi_i\}_{i=1}^\infty$ on $(X, \mu)$ be isomorphic to the standard one, it is necessary and sufficient that, for every measurable $A \subset X$ and every $\gamma > 0$, there exist an $n$ and an element $\alpha_n \in S_n(\{0; 1\})$ such that

$$\int_X \rho_n([\mathcal{P}_{\{\xi_i\}_{i=1}}{}^n \chi_A](x), \alpha_n)\, d\mu < \gamma;$$

here $\chi_A$ is the characteristic function of the set $A$; $\rho_n$ is the metric on $S_n$, defined inductively by

$$\rho_n(\alpha, \alpha') = \inf \int_X \rho_{n-1}(g(x), g'(x))\, d\mu,$$

where the infimum is taken over all such pairs $g, g' \in L(X; S_{n-1})$ that $g\mu = \alpha$, $g'\mu = \alpha'$, and $\rho_0(\{0\}, \{1\}) = 1$.*

Commenting on Theorem 2, we note that its condition requires almost constancy of $\mathcal{P}_{{\{\xi_i\}_1}^n}\chi_A$, while the constancy of the latter function means measurability of $A$ with respect to a well-coordinated with $\{\xi_i\}_1^n$, independent complement $\zeta$ to $\xi$ (analogous to the partition of $[0,1]$ into the intervals $[p/2^n,(p+1)/2^n]$, $p=0,\dots,2^n-1$, which is an independent complement to $\xi_n^{(0)}$).

**Corollary 1.** An ergodic action with discrete spectrum of the group $D$ of all roots of unity of degree $2^n,\ n=1,\dots,$ on an interval generates a standard dyadic sequence $\{\xi_n\}$ (here $\xi_n$ is the partition into the trajectories of the subgroup of roots of degree $2^n$).

It suffices to observe that there exists a multiplicative basis on the interval from whose elements one can form the system $\xi$.

**3°. Combinatorial lemmas.** Let us clarify in more detail the structure of the range of values of $\mathcal{P}$ for a dyadic sequence. Since almost every element of $\xi_1$ has two points of equal conditional measure, the values of $\mathcal{P}_{\xi_1}f,\ f\in L(X;K)$, will be measures on $K$ of the form ${}^1/_2(\delta_{k_1}+\delta_{k_2});\ k_{1,2}\in K$. Denote their set by $S^d(K)\subset S(K)$. Analogously,

$$\mathcal{P}_{{\{\xi_i\}_1}^n}f\in S_n^d(K),$$

where

$$S_n^d(K)=S^d(S_{n-1}^d(K)).$$

Let $I_N$ be the set of vertices of the $N$-dimensional unit cube; $H_n$ the group of permutations of the $2^n$ symbols $1,2,\dots,2^n$, constructed inductively as follows: $H_1=\mathfrak{S}_2$; $H_n$ is the minimal subgroup of $\mathfrak{S}_{2^n}$ containing the group $H_{n-1}\times H_{n-1}$ (where the factors act respectively on the symbols $1,2,\dots,2^{n-1}$ and $2^{n-1}+1,\dots,2^n$), as well as the permutation

$$(1,2^{n-1}+1)\times(2,2^{n-1}+2)\cdots(2^{n-1},2^n).$$

The action of $H_n$, applied to the coordinates of the $2^n$-dimensional cube, induces its action on $I_{2^n}$, denoted by the same letter $H_n$. Introduce on $I_N$ the Hamming metric

$$r_N(q,q')=\frac{1}{N}\sum_{i=1}^{N}|q_i-q_i'|$$

and transfer it to $I_{2^n}/H_n$ by the rule

$$\tilde r_{2^n}(Q,Q')=\inf r_{2^n}(q,q'),\quad q\in Q,\ q'\in Q'.$$

**Lemma 2.** $(S_n^d(\{0;1\}),\rho_n)$ is canonically isometric to $(I_{2^n}/H_n,\tilde r_{2^n})$.

Let $m_N$ be the uniform distribution on $I_N$.

**Lemma 3.** In order that a sequence of sets $\Gamma_N\subset I_N,\ N=1,\dots,$ have the property that, for every $\gamma>0$,

$$\lim_{N\to\infty} m_N\{q\in I_N:\ r_N(q,\Gamma_N)<\gamma\}=1,$$

it is necessary that

$$|\Gamma_N| \sim 2^{N(1-\beta_N)}, \qquad \beta_N \to 0.$$

**Lemma 4.** The longest trajectory of $H_n$ in $I_{2^n}$ has $2^{3/4 \cdot 2^n - 1}$ points ($n \geq 2$).

**Corollary 2.** The sequence $\tilde{m}_{2^n}$ of projections of the measures $m_{2^n}$ onto $I_{2^n}/H_n$ does not converge weakly to a $\delta$-measure, i.e. for every sequence $\{Q_n\}$, $Q_n \in I_{2^n}/H_n$, $n = 1, \dots$, and $\gamma > 0$,

$$\overline{\lim}\, \tilde{m}_{2^n}\{Q \in I_{2^n}/H_n : \tilde{r}_{2^n}(Q, Q_n) < \gamma\} < 1.$$

**4°. Examples.** Let $G$ be a countable group; $F(G)$ the space of all functions on $G$ with values 0 or 1; $\nu$ the measure on $F(G)$, the countable product of the measures $({}^1/_2, {}^1/_2)$ on $\{0; 1\}$; $G$ acts as a group of automorphisms on $(F(G), \nu)$:

$$g \mapsto V_g, \qquad (V_g f)(h) = f(gh), \quad f \in F(G);$$

($\{V_g\}$ is the generalized Bernoulli group).

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* This metric coincides with the Kantorovich-Rubinstein metric ([11]).

a) Let $D$ be the group of all roots of unity of degree $2^n$, $n = 1, \dots$. Let $\varepsilon_n$ be the partition $(F(D), \nu)$ into the trajectories of the action of the (periodic) automorphism $V_g$, where $g^n$ is a primitive root of degree $2^n$.

The dyadic sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ is not isomorphic to the standard one.

**Proof.** Let $A = \{f \in F(D) : f(e) = 1\}$. It is verified that the image of $\nu$ under the mapping

$$\mathcal{P}_{\{\varepsilon_i\}1^n} \chi_A : \; F(D) \to S_n^d(\{0; 1\}) = I_{2^n/H_n}$$

is the measure $\widetilde{m}_{2^n}$, and it follows from Corollary 2 that the conditions of Theorem 2 are not satisfied. The example proving Theorem 1 has been constructed.

b) We construct a Markov process—a random walk along the trajectories of a dynamical system.*

Let $W_2$ be the free group with two generators $w_1$ and $w_2$, $T_i = V_{w_i}$, $i = 1, 2$, automorphisms of $(F(W_2), \nu)$ (see above). The Markov process, and with it the Markov endomorphism $R$, is defined as follows: the set of states is $F(W_2)$, the initial distribution is $\nu$; the transition probability is

$$p(f, E) = \frac{1}{2},$$

if $T_1 f \in E$ or $T_2 f \in E$, and 0 in the remaining cases. This endomorphism is exact (the process is regular), and its entropy is easily computed:

$$h(R) = H(\varepsilon \mid R^{-1}\varepsilon) = 1.$$

The constructed Markov endomorphism $R$ is not a factor-endomorphism of a Bernoulli endomorphism with two states of equal measure.

**Proof.** The dyadic sequence $\{R^{-n}\varepsilon\}_{n=1}^{\infty}$ is not isomorphic to the standard one (see example a)); for $A$ one must take the set of trajectories $x = (x_n)_{n\leqslant 0}$ for which $x_0(e) = 1$, where $e$ is the identity of $W_2$.**

5°. **Remarks.** 1. The existence of a continuum of pairwise nonisomorphic dyadic sequences is established with the help of the notion of the entropy of a sequence of partitions.

2. For lack of space, applications to the theory of nonmeasurable (approximately measurable) partitions are not presented here.

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## REFERENCES

1. M. Rosenblatt, J. Math. Mech., 8, 5, 665 (1959).
2. M. Rosenblatt, ibid., 9, 6, 945 (1960).
3. V. A. Rokhlin, UMN, 4, 2, 57 (1949).
4. A. M. Vershik, Functional Analysis, 2, 3, 17 (1968).
5. R. M. Belinskaya, ibid., 2, 3, 4 (1968).
6. V. A. Rokhlin, UMN, 5 (137), 3 (1967).
7. D. A. Vladimirov, Boolean Algebras, "Nauka," 1969.
8. O. V. Guseva, Vestn. LGU, No. 1, issue 1, 14 (1965).
9. S. Kakutani, Proc. 2-nd Berkeley Sympos. Math. Statist. and Prob., 1951, p. 247.
10. V. I. Oseledets, Probability Theory and Its Applications, 10, 3 (1965).
11. L. V. Kantorovich, G. Sh. Rubinshtein, Vestn. LGU, 7, No. 2, 52 (1958).

* Similar Markov processes were considered in (9)—ergodicity conditions—and in (10)—the $K$-property.

** The constructed Markov process has a paradoxical property, which, in a free formulation, looks as follows. Suppose that the process of the development of mathematics is the same as the constructed process; if the editorial board every year, starting from $-\infty$, publishes a volume of *Itogi nauki* containing only everything substantially new discovered in that year, then reading all these volumes does not make it possible to reconstruct completely the picture of mathematics in any given year. (Of course, no nontrivial discoveries occurred at $-\infty$.)

*Note: Figure translations are in progress. See original paper for figures.*