APPROXIMATE FORMULAS FOR THE POLARIZABILITY TENSOR AND CAPACITANCE OF BODIES OF ARBITRARY SHAPE AND SOME APPLICATIONS

MATHEMATICAL PHYSICS

1970

SovietRxiv

Abstract

Full Text

UDC 517.94:538.574.4:59,55

MATHEMATICAL PHYSICS

A. G. RAMM

APPROXIMATE FORMULAS FOR THE POLARIZABILITY TENSOR AND CAPACITANCE OF BODIES OF ARBITRARY SHAPE AND SOME APPLICATIONS

(Presented by Academician V. A. Fock, 4 V 1970)

In Sec. 1 a formula is given for the polarizability tensor of a homogeneous dielectric body D with constant ε_i, placed in a medium with constant ε_l. In Sec. 2 a formula is given for the electrostatic capacitance of a conductor of arbitrary shape. In Sec. 3 applications are considered.

1. We shall denote the polarizability tensor of the body by $\alpha_{ij}(\gamma)$, where $\gamma = (\varepsilon_i - \varepsilon_l)/(\varepsilon_i + \varepsilon_l)$. Let V be the volume of the body D. Then, when it is placed in an external constant electrostatic field E, this body acquires a dipole moment

$$P_i = \sum_{j=1}^{3} \alpha_{ij}(\gamma) V E_j. \tag{1}$$

By Γ we denote the surface of the body D, $N_i(s)$ the component of the normal to the surface Γ, passing through the point s, and $r_{st} = |s - t|$ the distance between the points s and t.

Put

$$\alpha_{ij}^{(n)}(\gamma) = \frac{2}{V} \sum_{m=0}^{n} M_{ij}^{(m)} \frac{(-1)^m}{(2\pi)^m} \frac{\gamma^{n+2} - \gamma^{m+1}}{\gamma - 1}, \tag{2}$$

where

$$M_{ij}^{(0)} = V\delta_{ij}, \qquad M_{ij}^{(1)} = \iint_{\Gamma\Gamma} \frac{N_i(s)N_j(t)}{r_{st}}\, ds\, dt, \tag{3}$$

$$M_{ij}^{(m)} = \iint_{\Gamma\Gamma} ds\, dt\, N_i(s) N_j(t) \int_\Gamma \cdots \int_\Gamma \frac{1}{r_{st_{m-1}}} \psi(t_1, t)\psi(t_2, t_1) \cdots$$

$$\cdots \psi(t_{m-1}, t_{m-2})\, dt_1 \dots dt_{m-1}. \tag{3a}$$

Here the notation

$$\psi(t, s) = \frac{\partial}{\partial N_t} \frac{1}{r_{ts}}. \tag{4}$$

has been used.

The quantity $\alpha_{ij}^{(n)}(\gamma)$ is the n-th approximation to the tensor $\alpha_{ij}(\gamma)$.

Theorem 1. The estimate holds

$$\left|\alpha_{ij}^{(n)}(\gamma) - \alpha_{ij}(\gamma)\right| \leq A q^{n+1}, \tag{5}$$

in which $0 < q < 1$. The constants A and q are determined only by the shape of the surface and by the number γ.

Remark 1. For a sphere $A = 1,\ q = 1/3$.

Remark 2. For $\varepsilon_i = \infty,\ \gamma = 1$, which corresponds to ideal conductivity of the body D, formula (2) takes the form:

$$\alpha_{ij}^{(n)} = \frac{2}{V} \sum_{m=0}^{n} M_{ij}^{(m)} \frac{(-1)^m}{(2\pi)^m} (n + 1 - m). \tag{6}$$

For $\varepsilon_i = 0,\ \gamma = -1$, which corresponds to an ideal magnetic material, formula (2) takes the form

$$\beta_{ij}^{(n)} = -\frac{1}{V} \sum_{m=0}^{n} M_{ij}^{(m)} \frac{1 + (-1)^{m+n}}{(2\pi)^m}. \tag{7}$$

The magnetic moment of the body D in a constant external field H is computed by the formula

$$M_i = \sum_{j=1}^{3} \beta_{ij} V H_j. \tag{1a}$$

2. Denote by C the electrostatic capacitance of the conductor D. Introduce the quantity

$$C^{(n)} = 4\pi S^2 \Bigg/ \left[\left(-\frac{1}{2\pi}\right)^n \iint_{\Gamma\Gamma} \frac{ds\,dt}{r_{st}} \int_\Gamma \cdots \int_\Gamma \psi(t, t_n)\psi(t_n, t_{n-1}) \cdots \psi(t_2, t_1)\, dt_1 \cdots dt_n \right]. \tag{8}$$

Here S is the area of the surface Γ, and the function $\psi(t, s)$ is defined by formula (4). The quantity $C^{(n)}$ may be regarded as the n-th approximation to the capacitance of the conductor.

Theorem 2. *The following estimate holds:*

$$\left|C^{(n)} - C\right| \le A_1 q^{n+1}, \tag{9}$$

where $0 < q < 1$*. The constants* A_1*,* q *are determined only by the shape of the surface and by the number* γ*.*

Remark 3. The number q in Theorems 1 and 2 is the same.

Remark 4. For $n = 0$, formula (8) takes a particularly simple form:

$$C^{(0)} = 4\pi S^2 \Bigg/ \iint_{\Gamma\Gamma} \frac{ds\,dt}{r_{st}}. \tag{10}$$

Formula (10) is essentially an analytic expression of the well-known empirical Howe rule for computing capacitance.

The proofs of Theorems 1 and 2 are based on the methods developed in papers ([1]).

3. a) The scattering amplitude on a small body D, under the assumption that magnetic dipole radiation may be neglected, has the form

$$f = b[\nu[P, \nu]], \qquad b = k_0^2 / 4\pi\varepsilon_0. \tag{11}$$

Here ε_0 is the dielectric constant of the medium, k_0 is the wave number, ν is the unit vector directed toward the observation point, and P is the dipole moment of the small body in a constant electrostatic field E, which coincides with the value of the electric vector of the incident electromagnetic field at the location of the small body. We note that if the dimensions of the body are small in comparison with the wavelength of the incident field, then the position of the body may be characterized by the coordinates of a single point.

The problem that interests us is the following. Is it possible to compute the vector of the primary field E at the location of the small body if the scattering amplitude on this body is known? The answer to this question is affirmative. For the proof, consider the linear system with a self-adjoint matrix:

$$\sum_{m=1}^{3} a_{im} E_m = f_i, \qquad 1 \le i \le 3; \qquad a_{im} = \sum_{j=1}^{3} bV(\delta_{ij} - \nu_i \nu_j) d_{jm}. \tag{12}$$

The tensor a_{ij} is determined by the geometry of the body and can be computed by formula (2). It turns out that system (12) has rank 2. The adjoint homogeneous system (12) has the eigenvector $\mathbf{v}$. The free term satisfies the condition $(f, \mathbf{v}) = 0$, where the parentheses denote the scalar product. This condition guarantees the solvability of system (12). The computational algorithm for finding the primary field E reduces to the following:

1) Find the solution G_1 of system (12) for $\mathbf{v} = \mathbf{v}_1$, $f_i = f_i(\mathbf{v}_1)$, where $\mathbf{v}_1$ is an arbitrary unit vector satisfying the condition $(G_1, \mathbf{v}_1) = 0$. Such a solution exists and is unique.
2) Find the solution G_2 of system (12) for $\mathbf{v} = \mathbf{v}_2$, $f_i = f_i(\mathbf{v}_2)$, where $\mathbf{v}_2$ is any unit vector perpendicular to the unit vector $\mathbf{v}_1$, satisfying the condition $(G_2, \mathbf{v}_2) = 0$.
3) Find E from the formula $E = G_1 + (G_2, \mathbf{v}_1)\mathbf{v}_1$, or from the formula $E = G_2 + (G_1, \mathbf{v}_2)\mathbf{v}_2$.

Indeed, $G_j = E - (E, \mathbf{v}_j)\mathbf{v}_j,\ j = 1, 2$. Therefore

$$G_1 + (E, \mathbf{v}_1)\mathbf{v}_1 = G_2 + (E, \mathbf{v}_2)\mathbf{v}_2.$$

Hence the equalities $(G_2, \mathbf{v}_1) = (E, \mathbf{v}_1)$, $(G_1, \mathbf{v}_2) = (E, \mathbf{v}_2)$ follow, which lead to formulas 3) of the algorithm.

It is not difficult to construct an algorithm for computing the field E from the values of the scattered-field vector measured in any two nonparallel directions $\mathbf{v}_1, \mathbf{v}_2$ (not necessarily perpendicular).

b) With the aid of Theorem 1 one can write the principal term of the scattering characteristic for a body small in comparison with the wavelength of the incident field. It suffices to substitute, in the well-known expression (see, for example, (16), p. 1192),

$$f_E = \frac{k_0^2}{4\pi} \left\{ \frac{1}{\varepsilon_0} [\mathbf{v}[P, \mathbf{v}]] - \sqrt{\frac{\mu_0}{\varepsilon_0}} [\mathbf{v}, M] \right\} \tag{13}$$

instead of P and M their values according to formulas (1), (1a). In formula (13), ε_0, μ_0 are the parameters of the medium, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, $\mathbf{v}$ is the unit vector directed toward the observation point, and P, M are the dipole electric and magnetic moments acquired by the body in static electric E and magnetic M fields. The fields E, H coincide with the values of the electric and, respectively, magnetic vector of the incident field at the location of the small body.

c) In the book (^{2}) the theory of light scattering by small particles, mainly of ellipsoidal shape, is presented. The results of item 1 make it possible to develop this theory for particles of arbitrary shape, which is of interest, for example, for atmospheric optics, colloid chemistry, astrophysics, and the theory of radio measurements. Scattering by large bodies of arbitrary shape was studied by V. A. Fok (^{3}).

d) Let us consider the scalar problem of determining the shape of a convex reflecting body from the scattering characteristic:

$$f(n, \mathbf{v}, k) = -\frac{1}{4\pi} \int_\Gamma \exp\{-ik(\mathbf{v}, s)\} \frac{\partial u}{\partial N}\, ds. \tag{14}$$

Here $n, \mathbf{v}$ are the directions of propagation of the incident and scattered waves, respectively. In the short-wavelength approximation

$$\left.\frac{\partial u}{\partial N}\right|_{\Gamma_-} = 0, \qquad \left.\frac{\partial u}{\partial N}\right|_{\Gamma_+} = 2\frac{\partial}{\partial N} \exp\{ik(n, s)\}.$$

Here Γ_+, Γ_- are the illuminated and shadow parts of the surface Γ. In this approximation formula (14) takes the form

$$f(n, \mathbf{v}, k) = \frac{k}{2\pi i} \int_{\Gamma_+} \exp\{-ik(\mathbf{v} - n)(l, s)\}(n, N_s)\, ds, \qquad l \equiv \frac{\mathbf{v} - n}{|\mathbf{v} - n|}. \tag{15}$$

Computing the integral as $k \to \infty$ by the method of stationary phase, we obtain

$$f(n, k, \mathbf{v}) \simeq -1/2\mathscr{K}(l), \qquad \mathscr{K}(l) = 1/\sqrt{R_1 R_2}. \tag{16}$$

This formula can be derived in another way, proceeding from the general results of (^{3}). Here $\mathscr{K}(l)$ is the Gaussian curvature of the surface at the point whose normal coincides with the unit vector l. By Minkowski' s theorem (^{4}), specifying the Gaussian curvature as a function of the external normal uniquely determines a convex surface. Consequently, knowing, for example, the function $f(n, -n, k)$ for all n and $k \to \infty$, one can reconstruct the shape of the reflecting body. An effective method of reconstruction is based on the following considerations.

From formula (15) there follows the equality

$$f(n, \nu, k) + f^*(-n, -\nu, k) = \frac{k}{2\pi i} \int_\Gamma \exp\{-ik(\nu, -n, s)\}$$

$$(n, N_s)\, ds = \frac{1 - (\nu, n)}{2\pi} k^2 \int_D \exp\{-ik|n - \nu|(l, y)\}\, dy. \tag{17}$$

Using the asymptotic formula ((^{5}), p. 42)

$$\int_D e^{ik(l,y)}\,dy = -\frac{4\pi}{\mathcal{K}(l)}\,\frac{\cos\{ka(l)\}}{k^2}\left(1+O\left(\frac{1}{\sqrt{k}}\right)\right), \qquad k\to+\infty, \tag{18}$$

where $a(l)$ is the half-width of the convex centrally symmetric body D in the direction l, we obtain from formula (17), as $k\to\infty$, the following asymptotic equality, calculated in a coordinate system whose origin is placed at the center of symmetry of the body D, and whose z-axis is directed along the vector l:

$$f(n,\nu,k)+f^*(-n,-\nu,k) = -\cos\{k_1 a(l)\}/\mathcal{K}(l),$$

$$k_1 = k|n-\nu|, \quad k\to\infty. \tag{19}$$

Expression (19) determines the function $a(l)$. Knowing this function, it is not difficult to write the parametric equation of the surface Γ (([4]), p. 168):

$$x=\partial a/\partial\alpha, \qquad y=\partial a/\partial\beta, \qquad z=\partial a/\partial\gamma, \tag{20}$$

where α,β,γ are the projections of the unit vector l onto the Cartesian axes.

We note that the function $a(l)$—the support function of a convex surface—is a homogeneous function of α,β,γ of order 1 ([4]).

In conclusion I express my sincere gratitude to Academician V. A. Fock for his interest in the work and valuable advice.

Leningrad Institute
of Precision Mechanics and Optics

Received
21 IV 1970

References

[1] A. G. Ramm, a) DAN, 186, No. 1, 62 (1969); b) Radiofizika, Izv. vyssh. uchebn. zaved., 12, No. 8, 1185 (1969).
[2] G. van de Hulst, *Light Scattering by Small Particles*, Moscow, 1961.
[3] V. A. Fock, a) Izv. AN SSSR, ser. fizich., 10, No. 2, 171 (1946); b) UFN, 36, No. 3, 308 (1948).
[4] W. Blaschke, *Circle and Sphere*, Moscow, 1967.
[5] I. M. Gelfand, M. I. Graev, N. Ya. Vilenkin, *Integral Geometry and Representation Theory*, Moscow, 1962.
[6] A. G. Ramm, a) Radiotekhnika i elektronika, 10, No. 11, 2062 (1965); b) III All-Union Symposium on Field Diffraction. Abstracts of Reports, "Nauka," 1964, p. 143.

Note: Figure translations are in progress. See original paper for figures.