QUASISYMMETRIC CONFORMALLY EUCLIDEAN SPACES AND A MATHEMATICAL MODEL OF THE SPACE-TIME CONTINUUM AS A WHOLE

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Abstract

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QUASISYMMETRIC CONFORMALLY EUCLIDEAN SPACES AND A MATHEMATICAL MODEL OF THE SPACE-TIME CONTINUUM AS A WHOLE

(Presented by Academician P. S. Novikov on 20 VI 1969)

1. Let g_{ij} be the metric tensor, R_{ij} the Ricci tensor, and R the scalar curvature of a Riemannian manifold. Quasisymmetric conformally Euclidean spaces of dimension $n \geq 4$ are defined by the conditions: $C^l_{ij,k} = 0$, $T_{ljk} = 0$, where $C^l_{ij,k}$ is the Weyl conformal-curvature tensor ([1,3]), and

$$T_{ljk} = \nabla_l T_{jk} - \frac{n-2}{2(1-n)(n+2)} \Delta({}^l T g_{jk}) + \frac{3n-4}{2n(1-n)} \nabla_l T g_{jk},$$

$$T_{jk} = -\frac{1}{\chi}\left(R_{jk} - \frac{2}{n} R g_{jk} + \Lambda g_{jk}\right), \qquad T = g^{jk} T_{jk} = \frac{1}{\chi}(R - n\Lambda)$$

$$(\chi, \Lambda = \text{const}).$$

In special conformally Euclidean coordinates

$$ds^2 = \frac{1}{U} \overset{\circ}{g}_{ij} dx^i dx^j,$$

$$\overset{\circ}{g}_{ij} = \overset{\circ}{g}_{ji} - \text{const}, \quad \det \|\overset{\circ}{g}_{ij}\| \neq 0,$$

where

$$U = a(\overset{\circ}{g}_{ij} x^i x^j)^2 + b_i x^i \overset{\circ}{g}_{jk} x^j x^k + \overset{\circ}{g}_{ki} L^k_j x^i x^j + d_i x^i + e$$

$$= a\langle x, x\rangle^2 + \langle b, x\rangle\langle x, x\rangle + \langle Lx, x\rangle + \langle d, x\rangle + e$$

$$(a, b_i, L_j^k, d_k, e - \text{const}, \quad \overset{\circ}{g}_{ki} L_j^k = \overset{\circ}{g}_{kj} L_i^k).$$

2. The spaces under consideration will be symmetric if and only if $R = \text{const}$ ([9]). The latter, in turn, is equivalent to the system

$$a\lambda = \frac{n+2}{8}\langle b, b\rangle, \qquad e\lambda = \frac{n+2}{8}\langle d, d\rangle, \qquad L^*|b\rangle = (n+2)a|d\rangle,$$

$$L^*|d\rangle = (n+2)e|b\rangle,$$

$$L^*\left[L^* - \lambda\left(\frac{|b\rangle\langle b|}{\langle b, b\rangle} + \frac{|d\rangle\langle d|}{\langle d, d\rangle} - E\right)\right] = \left[(n+2)^2 ae - \frac{(n+2)^2}{8}\langle b, d\rangle\right] E,$$

where

$$\lambda = \operatorname{tr} L - \frac{R}{2(1-n)}, \qquad L^* = \frac{n+2}{2}L - \lambda E, \qquad E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Symmetric conformally Euclidean spaces are especially rich in good geometric properties, since they admit a transitive group of motions ([2]) and possess a conformal interpretation. If $R_{ij,kl} \neq 0$, then, in the case of positive definiteness of the fundamental quadratic form, the metric of such spaces can be brought to the form

$$S^2 = \frac{dx^{1\,2} + dx^{2\,2} + \ldots + dx^{n\,2}}{A^2\left[(x^{1\,2} + x^{2\,2} + \ldots + x^{n\,2})^2 + 2(-x^{1\,2} - x^{2\,2} - \ldots - x^{\nu\,2} + x^{\nu+1\,2} + \ldots + x^{n\,2}) + 1\right]}$$

$$(A > 0).$$

Hence it is seen that, for $\nu \neq 0$, the n-symmetric conformally Euclidean spaces differ from spaces of constant curvature.

The equation of the absolute

$$(x^{1^2} + x^{2^2} + \ldots + x^{n^2})^2 + 2(-x^{1^2} - x^{2^2} - \ldots - x^{\nu^2} + x^{\nu+1^2} + \ldots + x^{n^2}) + 1 = 0$$

selects in n-dimensional real space a $(\nu - 1)$-dimensional sphere

$$x^{1^2} + x^{2^2} + ... + x^{\nu^2} = 1,$$

$$x^{\nu+1^2} + x^{\nu+2^2} + ... + x^{n^2} = 0.$$

Assigning to ν the values $0, 1, 2, \dots, n$, we obtain the real parts of the absolutes of all symmetric conformally Euclidean spaces.

3. In constructing a mathematical model of the space-time continuum as a whole in the general theory of relativity, of special interest are those quasi-symmetric conformally Euclidean spaces which can be considered within the framework of general, nonstationary Friedmann models ([3,6]), with a cosmological term Λ different from zero. Here the mass tensor is taken in the general hydrodynamical form ([3,5])

$$T^{ik} = \left[\rho^* + \frac{1}{c^2}(\rho^*\Pi + p)\right]\tau^i\tau^k - \frac{p}{c^2}\overset{\circ}{g}{}^{ik}, \qquad \overset{\circ}{g}{}^{ik} = \overset{\circ}{g}{}_{ik} = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & +1 \end{pmatrix},$$

where ρ^* is the density of the invariant part of the rest mass, p is the pressure,

$$\Pi = \int_0^p \frac{dp}{\rho^*} - \frac{p}{\rho^*}$$

is the potential energy of unit mass, c is the speed of light, and τ^i are the components of the four-velocity vector.

The corresponding metric, under the natural additional requirement ($\rho^* > 0$), takes the form:

$$ds^2 = \frac{-dx^{1^2} - dx^{2^2} - dx^{3^2} + dx^{4^2}}{A^2\{[-x^{1^2} - x^{2^2} - x^{3^2} + x^{4^2} - (1-k^2)]^2 + 4x^{4^2}\}} \qquad (0 < A,\ 0 < k < 1).$$

Passing from conformally Euclidean coordinates to other, specially chosen coordinates, we obtain:

$$ds^2 = c^2dt^2 - \frac{1}{4A^2\,\mathrm{dn}^2(2Act, k)}\,\frac{dx^{1^2} + dx^{2^2} + dx^{3^2}}{\left[1 + \frac{1}{4}(x^{1^2} + x^{2^2} + x^{3^2})\right]^2}.$$

Here c is the speed of light, t is time, and $\mathrm{dn}(2Act, k)$ is the last of the three Jacobi elliptic functions $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ of modulus k.

Fig. 1

Figure 1: Fig. 1

We note that for $k = 0$ the model under consideration gives the stationary world of A. Einstein ([4]).

The following quantities are easily expressed in terms of A and k:

1) Cosmological term

$$\Lambda = 4A^2(1 - 3k^2).$$

2) Pressure

$$p = -\frac{20A^2}{\varkappa} k^2 \frac{\operatorname{cn}^2(2Act, k)}{\operatorname{dn}^2(2Act, k)}, \qquad p_{\min} = -\frac{20A^2k^2}{\varkappa}, \qquad p_{\max} = 0.$$

3) Mass of the idealized Universe

$$M = 2\pi^2/\varkappa c^2 A(1 - k^2)^{3/2}.$$

4) Mean mass density

$$\rho^* = \frac{8A^2}{\varkappa c^2}(1 - k^2)^{-3/2} \operatorname{dn}^3(2Act, k),$$

$$\rho^*_{\min} = \frac{8A^2}{\varkappa c^2}, \qquad \rho^*_{\max} = \frac{8A^2}{\varkappa c^2}(1 - k^2)^{-3/2}.$$

5) Radius

$$B = \frac{1}{2A} \frac{1}{\operatorname{dn}(2Act, k)}, \qquad B_{\min} = \frac{1}{2A}, \qquad B_{\max} = \frac{1}{2A}(1 - k^2)^{-1/2}.$$

6) The period of variation of the radius as a function of time

$$\tau = \frac{1}{Ac} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.$$

Fig. 1 gives a general idea of the character of the variation of the radius of the idealized Universe with time.

Fig. 1

4. The numerical values of the parameters A and k (and also the value of t corresponding to our epoch) can be found from experimental data:

1) The mean mass density of the Universe in our epoch (5)

$$\rho^*_{\text{present}} = 4 \cdot 10^{-29}\ \text{g/cm}^3.$$

2) Hubble' s constant (7)

$$H_{\text{present}} = 2.43 \cdot 10^{-18}\ \text{sec}^{-1}.$$

3) The acceleration parameter (8)

$$K_{\text{present}} = -qH^2.$$

Here $0.5 < q < 1.5$.

4) The gravitational constant

$$G = 6.67 \cdot 10^{-8}\ \text{cm}^3/\text{g} \cdot \text{sec}^2$$

and the speed of light

$$c = 2.997 \cdot 10^{10}\ \text{cm/sec}.$$

The parameters A and k (and also the value of the time t) are related to the mean mass density ρ^* and to the quantities $H = B'_t/B$ and $K = B''_{tt}/B$ by the relations

$$\rho^* = \frac{8A^2}{\varkappa c^2}(1-k^2)^{-3/2}\,\text{dn}^3(2Act, k),$$

$$H = \frac{2k^2cA^2\,\text{sn}(2Act,k)\,\text{cn}(2Act,k)}{\text{dn}(2Act,k)},$$

$$K = 4A^2k^2c^2 + 8A^2c^2(\text{dn}^2(2Act,k)+1) + 2H^2.$$

Substituting into this system the numerical values ρ^*_{present}, H_{present}, K_{present}, found experimentally, as well as the values of the speed of light c and the constant $\varkappa = 8\pi G/c^4$, one can, using the capabilities of modern computing technology, find the parameters A and k and the present value of the time t_{present}. Next, using formulas 1)–6) of item 3, we obtain the values of all the remaining quantities that give the basic physical characteristics of the model.

The corresponding calculations were carried out on the BESM-4 computer by V. Topunov.

Since the results of the calculations depend substantially on q, a table arises. However, it is necessary to note especially that, for the selected values

$$\rho^*_{\text{present}} = 4 \cdot 10^{-29} \text{ g/cm}^3$$

and

$$H_{\text{present}} = 2.43 \cdot 10^{-18} \text{ sec}^{-1},$$

in the interval (1/2, 3/2) there exists such a value

$$q = Q = 1.31\,067\,645\ldots,$$

as it tends to which $k \to +1$, and the metric of the quasisymmetric conformally Euclidean model takes the form:

$$ds^2 = c^2dt^2 - \frac{1}{4A^2}\,\mathrm{ch}^2(2Act)\,\frac{dx^{1\,2} + dx^{2\,2} + dx^{3\,2}}{\left[1 + \frac{1}{4}\left(x^{1\,2} + x^{2\,2} + x^{3\,2}\right)\right]^2}.$$

Hence

$$B = \frac{1}{2A}\,\mathrm{ch}(2Act).$$

Table 1

Fragments of a table of values of the principal parameters of the quasisymmetric conformally Euclidean model of the space-time continuum as a whole

q	0.5	1	1.3	1.31067645...	1.31067645...	1.31067645 ...
A^2 [cm^{-2}]	$4.42 \cdot 10^{-57}$	$5.10 \cdot 10^{-57}$	$5.42 \cdot 10^{-57}$	$5.43 \cdot 10^{-57}$	$5.43 \cdot 10^{-57}$	$5.43 \cdot 10^{-57}$
$1-k^2$	$3.2 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$7.0 \cdot 10^{-4}$	10^{-9}	$8.2 \cdot 10^{-20}$	$1.8 \cdot 10^{-24}$
Λ [cm^{-2}]	$-3.37 \cdot 10^{-56}$	$-3.97 \cdot 10^{-56}$	$-4.33 \cdot 10^{-56}$	$-4.34 \cdot 10^{-56}$	$-4.34 \cdot 10^{-56}$	$-4.34 \cdot 10^{-56}$
$\frac{P_{\text{present}}}{c^2}$ [g $\cdot$ cm^{-3}]	$-2.29 \cdot 10^{-29}$	$-1.93 \cdot 10^{-29}$	$-1.77 \cdot 10^{-29}$	$-1.76 \cdot 10^{-29}$	$-1.76 \cdot 10^{-29}$	$-1.76 \cdot 10^{-29}$
$\frac{P_{\min}}{c^2}$ [g $\cdot$ cm^{-3}]	$-4.59 \cdot 10^{-29}$	$-5.38 \cdot 10^{-29}$	$-5.81 \cdot 10^{-29}$	$-5.83 \cdot 10^{-29}$	$-5.83 \cdot 10^{-29}$	$-5.83 \cdot 10^{-29}$

	q	0.5	1	1.3	1.31067645...	1.31067645...	1.31067645 ...
$B_{present}$ [cm]		$3.3 \cdot 10^{28}$	$4.3 \cdot 10^{28}$	$2.1 \cdot 10^{29}$	$1.8 \cdot 10^{32}$	$2.0 \cdot 10^{37}$	$4.2 \cdot 10^{39}$
[billion light years]		$3.5 \cdot 10$	$4.5 \cdot 10$	$2.2 \cdot 10^2$	$1.9 \cdot 10^5$	$2.1 \cdot 10^{10}$	$4.4 \cdot 10^{12}$
B_{min} [cm]		$7.5 \cdot 10^{27}$	$7.0 \cdot 10^{27}$	$6.8 \cdot 10^{27}$	$6.8 \cdot 10^{27}$	$6.8 \cdot 10^{27}$	$6.8 \cdot 10^{27}$
[billion light years]		7.9	7.4	7.2	7.2	7.2	7.2
B_{max} [cm]		$4.2 \cdot 10^{28}$	$5.2 \cdot 10^{28}$	$2.6 \cdot 10^{29}$	$2.1 \cdot 10^{32}$	$2.4 \cdot 10^{37}$	$5.0 \cdot 10^{39}$
[billion light years]		$4.4 \cdot 10$	$5.5 \cdot 10$	$2.7 \cdot 10^2$	$2.2 \cdot 10^5$	$2.5 \cdot 10^{10}$	$5.3 \cdot 10^{12}$
ρ^*_{min} [g • cm^{-3}]		$1.90 \cdot 10^{-29}$	$2.19 \cdot 10^{-29}$	$2.33 \cdot 10^{-29}$	$2.33 \cdot 10^{-29}$	$2.33 \cdot 10^{-29}$	$2.33 \cdot 10^{-29}$
ρ^*_{max} [g • cm^{-3}]		$3.33 \cdot 10^{-27}$	$9.20 \cdot 10^{-27}$	$1.20 \cdot 10^{-24}$	$7.36 \cdot 10^{-16}$	1	10^7
M [g] (mass of an ideal universe)		$2.8 \cdot 10^{58}$	$6.2 \cdot 10^{58}$	$7.2 \cdot 10^{66}$	$4.6 \cdot 10^{69}$	$6 \cdot 10^{84}$	$6 \cdot 10^{91}$
$N = \frac{M}{M_0^*}$		$7.0 \cdot 10^{13}$	$1.6 \cdot 10^{14}$	$1.8 \cdot 10^{22}$	$1.2 \cdot 10^{25}$	$1.5 \cdot 10^{40}$	$1.5 \cdot 10^{47}$
τ [sec]		$1.56 \cdot 10^{18}$	$1.59 \cdot 10^{18}$	$2.26 \cdot 10^{18}$	$5.29 \cdot 10^{18}$	$1.05 \cdot 10^{18}$	$1.29 \cdot 10^{18}$
[billion years]		$4.9 \cdot 10$	$5.0 \cdot 10$	$7.2 \cdot 10$	$1.68 \cdot 10^2$	$3.33 \cdot 10^2$	$4.09 \cdot 10^2$
$t_{present}$ [sec]		$5.94 \cdot 10^{17}$	$6.40 \cdot 10^{17}$	$9.86 \cdot 10^{17}$	$2.58 \cdot 10^{18}$	$5.21 \cdot 10^{18}$	$6.40 \cdot 10^{18}$
[billion years]		$1.9 \cdot 10$	$2.0 \cdot 10$	$3.1 \cdot 10$	$8.2 \cdot 10$	$1.65 \cdot 10^2$	$2.03 \cdot 10^2$
$\frac{2t_{present}}{\tau}$		0.76	0.81	0.87	0.98	0.989	0.990

* The mass of our galaxy is $M_0 = 4 \cdot 10^{44}$ g.

The calculations carried out show that the quasisymmetric conformally Euclidean model of the space-time continuum as a whole that is under consideration leads to reasonable cosmological conclusions.

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Note: Figure translations are in progress. See original paper for figures.