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Abstract

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THE AVERAGING METHOD FOR NONSINUSOIDAL WAVES

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Recently, certain general results have been obtained for nonlinear wave processes close to stationary (propagating without deformation) plane waves in conservative systems described by Lagrangian-type equations ($^{1-3}$). In connection with problems of this and of a broader class, relating, for example, to nonconservative systems, it is evidently expedient to develop a more or less universal scheme of expansion in a small parameter characterizing the local deviation of the wave from a stationary one (as has already been done for one nonlinear equation of second order (^{4})).

In the present work the corresponding method of asymptotic expansion is proposed for an arbitrary system of nonlinear first-order partial differential equations.

Consider the system of equations

$$M(\tau, \rho, u, u_t, u_x, u_y, u_z) = \varepsilon F(\tau, \rho, u, u_t, u_x, u_y, u_z), \tag{1}$$

where $u(\mathbf{r}, t)$ is an N-dimensional vector function; M and F are sets of known nonlinear functions; t is time; $\mathbf{r}$ is a spatial coordinate; ε is a small parameter; $\tau = \varepsilon t$, $\rho = \varepsilon r$ are parameters characterizing the nonstationarity and inhomogeneity of the medium. With respect to M and F we assume only that they are sufficiently smooth functions of their arguments.

Let, for $\varepsilon = 0$ and constant τ and ρ, (1) have solutions in the form of stationary plane waves $u = U(\theta)$, $\theta = \omega t - \mathbf{k}\mathbf{r}$, which are determined from the system of ordinary differential equations

$$M(\tau, \rho, U, \omega U_\theta, -kU_\theta) = 0, \qquad \tau, \rho = \text{const} \tag{2}$$

and depend on $m \le N$ arbitrary constants of integration θ_0, $A = (A_2, \dots, A_m)$, and on the parameters $\tau, \rho, \omega, \mathbf{k}$. The frequency ω and the wave number $\mathbf{k}$ are connected with A by the dispersion relation $\omega = \omega(\mathbf{k}, A)$ and are chosen in such a way that U is a periodic function of θ with period 2π.

If (2) describes a conservative system, then periodic solutions correspond to trajectories filling some subspace in the phase space of solutions of (2) for a given ω or $\mathbf{k}$. For a nonconservative system the periodic solution $U(\theta)$, if it exists, is represented by an isolated trajectory in phase space; for it all A are fixed. The wave profile is determined by the concrete form of (2) and may differ strongly from a sinusoidal one.

We shall seek a solution of the original system of equations (1), close to $U(\theta)$, in the form of an asymptotic series in the small parameter ε*

$$u = U(\theta, A, \tau, \rho) + \sum_{n=1}^{\infty} \varepsilon^n u^{(n)}(t, \mathbf{r}); \tag{3a}$$

$$\omega(\tau, \rho) = \theta_t; \qquad \mathbf{k}(\tau, \rho) = -\nabla\theta, \tag{3b}$$

* We note that this method of constructing the asymptotic series differs from the one known for quasiharmonic waves with constant ω and $\mathbf{k}$ (^{5}).

where

$$A(\tau, \rho) = \sum_{n=0}^{\infty} \varepsilon^n A^{(n)}(\tau, \rho); \qquad \theta = \theta^{(0)}(t, \mathbf{r}, \tau, \rho) + \sum_{n=1}^{\infty} \varepsilon^n \theta^{(n)}(\tau, \rho). \tag{4}$$

Let us expand M and F in series in ε, taking (3) into account:

$$M = M^{(0)}(U, \omega U_\theta, -\mathbf{k}U_\theta) +$$

$$+\varepsilon \left\{ \frac{\partial M^{(0)}}{\partial U} u^{(1)} + \frac{\partial M^{(0)}}{\partial U_t}(U_\tau + u_t^{(1)}) + \frac{\partial M^{(0)}}{\partial \nabla U}(\nabla_\rho U + \nabla u^{(1)}) \right\} + \varepsilon^2 \ldots, \tag{5}$$

$$\varepsilon F = \varepsilon F^{(0)}(U, \omega U_\theta, -\mathbf{k}U_\theta) + \varepsilon^2 \ldots,$$

where $\nabla_\rho U = \partial U/\partial\rho$, and terms of the type $\dfrac{\partial M^{(0)}}{\partial U} U$ are the product of the square matrix $\dfrac{\partial M}{\partial U}(U, \omega U_\theta, -\mathbf{k}U_\theta)$ by the column U.

Substituting (3)–(5) into (1) and equating the coefficients of like powers of ε, we obtain, taking (2) into account,

$$\widehat{T}\left(\tau, \rho, \theta, \frac{\partial}{\partial t}, \nabla\right) u^{(n)} = H^{(n)}; \qquad n = 1, \dots, \tag{6}$$

where

$$\widehat{T} u^{(n)} = \frac{\partial M^{(0)}}{\partial U} u^{(n)} + \frac{\partial M^{(0)}}{\partial U_t} u_t^{(n)} + \frac{\partial M^{(0)}}{\partial \nabla U} \nabla u^{(n)}; \tag{7}$$

$$H^{(1)} = F^{(0)} - \frac{\partial W^{(0)}}{\partial U_t} U_\tau - \frac{\partial M^{(0)}}{\partial \nabla U} \nabla_\rho U; \qquad H^{(2)} = \dots \tag{8}$$

Consequently, the determination of the functions A and θ in any approximation is connected with finding a forced solution of a system of linear nonhomogeneous partial differential equations with periodic coefficients and a right-hand side periodic in θ. Naturally, the solution of (6) is also a function of θ (and of the parameters τ, ρ). Therefore its determination reduces to solving a system of ordinary differential equations with periodic coefficients, which is obtained from (6) by the substitution $\partial/\partial t = \omega \partial/\partial\theta$, $\nabla = -\mathbf{k}\partial/\partial\theta$. Then the solution $u^{(n)}$ can be represented in the form

$$u^{(n)} = Y \int_0^\theta Y^{-1} H^{(n)}\, d\theta', \tag{9}$$

where Y is the matrix composed of the vectors of a fundamental system of solutions of the variational equations $\widehat{T}(\tau, \rho, \omega, \mathbf{k}, \theta, \partial/\partial\theta)\psi = 0$, and Y^{-1} is the matrix inverse to Y.

Let us note that m particular vectors of the indicated system of solutions are determined through U [^6]:

$$Y_1 = \partial U/\partial\theta, \qquad Y_k = \partial U/\partial A_k, \quad k = 2, \dots, m. \tag{10}$$

It is easy to see that Y_1 is a periodic function of θ, while Y_k in the general case is representable as [^4,^6]

$$Y_k = U_{A_k} + \alpha_k \theta U_\theta, \tag{11}$$

where by U_A here and below is meant the derivative of U with respect to the explicitly entering parameter A, and α is a constant determined by the dependence of ω and $\mathbf{k}$ on A.

The remaining $N - m$ vectors of the fundamental system of solutions can, by Floquet' s theorem, be written in the form $Y_i = e^{\lambda_i \theta} f_i(\theta)$, with $f(\theta + 2\pi) \equiv f(\theta)$. Suppose that the characteristic exponents λ have no positive real part. In addition, we shall assume that if $\operatorname{Re}\lambda_l = 0$, then $\operatorname{Im}\lambda_l \neq 0, \pm 1, \ldots$. These conditions are usually used in the theory of oscillations as conditions for the absence of internal resonance [~7]. In this case all Y_i are bounded functions of θ, not in "resonance" with the family of solutions under consideration.

Representing Y and Y^{-1} in (9), after simple transformations we obtain

$$u^{(n)} = U_\theta \int_0^\theta d\theta' \left\{ U_\theta H^{(n)} + \sum_{k=2}^{m} \int_0^{\theta'} U_{A_k} H^{(n)}\, d\theta'' \right\} +$$

$$+ \sum_{k=2}^{m} U_{A_k} \int_0^\theta \{U_{A_k} H^{(n)}\}\, d\theta' + \sum_{i=m+1}^{N} e^{\lambda_i \theta} f_i \cdot \int_0^\theta e^{-\lambda_i \theta'} \{f_i^* H^{(n)}\}\, d\theta'. \qquad (12)$$

By virtue of the assumptions on the nature of λ, the last term in (15) is bounded for any θ. In order that $u^{(n)}$ be a bounded function of θ, it is necessary that the orthogonality conditions be satisfied:

$$\int_0^{2\pi} U_{A_k} H^{(n)}\, d\theta = 0, \qquad (13a)$$

$$\int_0^{2\pi} U_\theta H^{(n)}\, d\theta = 0, \qquad (13b)$$

$$n = 1, \ldots, \qquad k = 2, \ldots, m.$$

To the equations obtained it is necessary to add the relations following from (3b):

$$\nabla\omega + \partial\mathbf{k}/\partial t = 0, \qquad \operatorname{rot}\mathbf{k} = 0. \qquad (14)$$

It is convenient to put the equations of the first approximation into a somewhat different form. We multiply (2) by U_A and integrate with respect to θ (for constant τ, ρ). From the resulting relation we subtract (13a). As a result we obtain

$$\int_0^{2\pi} M(\tau, \rho, U, U_t, \nabla U)\, U_{A_k}\, d\theta = \varepsilon \int_0^{2\pi} F(\tau, \rho, U, \omega U_\theta, -\mathbf{k} U_\theta)\, U_{A_k}\, d\theta, \qquad (15a)$$

and analogously, from (2) and (13b), it follows that

$$\int_0^{2\pi} M(\tau, \rho, U, U_t, \nabla U)\, U_\theta \, d\theta = \varepsilon \int_0^{2\pi} F(\tau, \rho, U, \omega U_\theta, -\mathbf{k} U_\theta)\, U \, d\theta. \qquad (15b)$$

Hence the method of obtaining the equations of the first approximation directly from the original system (1) is clear.

The system (14)–(15) is complete with respect to the unknown functions $A, \omega, \mathbf{k}$. However, to determine the phase θ with accuracy up to ε, it is necessary to solve the equations of the second approximation. Such a situation is characteristic of nonisochronous oscillatory systems (7). It is clear that in the second approximation obtaining the solution is considerably more difficult, since it is necessary to know all the vectors of the fundamental system. Only in the case when the general solution (2) is known ($m = N$) are all Y_i expressed in terms of U by formulas (10).

It is not difficult to see that equations (14)–(15) are quasilinear and may be of either hyperbolic or elliptic type (the latter possibility is connected with the nonlinearity of the original system) (2). In the hyperbolic case there exists a system of real characteristics determining certain trajectories (rays) in the space $(\mathbf{r}, t)$. In this sense, the given method may be regarded as a generalization of geometrical optics to nonlinear media.

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Note: Figure translations are in progress. See original paper for figures.