# On Steady Capillary-Gravity Waves of Finite Amplitude Caused by a Pressure Periodically Distributed over the Surface of a Flow of Liquid of Infinite Depth

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## Abstract

## Full Text

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# On Steady Capillary-Gravity Waves of Finite Amplitude Caused by a Pressure Periodically Distributed over the Surface of a Flow of Liquid of Infinite Depth

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An exact solution is given of the problem considered here, by specifying the pressure on the surface by means of a certain infinite trigonometric series. The special case is also investigated in which the wavelength of the prescribed pressure coincides with the length of the steady free linear wave corresponding to the given flow velocity and to a constant pressure along the surface. The waves considered here cease to exist when the periodic part of the pressure distributed over the surface is identically set equal to zero, and the motion becomes a uniform flow. Such waves are called forced waves ($^1$). Here we briefly set forth the results obtained by us. An analogous problem for gravitational waves was considered by us earlier ($^2$). Steady, but free, capillary-gravity waves were also studied by us earlier ($^{3,4}$) by the Levi-Civita method, which reduces the problem to a nonlinear differential equation. Here the problem is reduced to the solution of a certain nonlinear integral equation.

Let us consider a plane-parallel steady motion of an ideal incompressible heavy liquid, bounded only above by a free surface, on which the pressure is $p = p_0' + p_0(x)$; here $p_0' = \text{const}$, while $p_0(x)$ is a prescribed periodic function of the horizontal coordinate $x$. Suppose that the flow moves from left to right with constant velocity $c$ at infinite depth. Since on the surface the pressure is a periodic function of $x$, the surface assumes the form of an immobile periodic wave in coordinates connected with a progressive wave having velocity $-c$. We show that forced waves exist for arbitrary finite values of the velocity $c$.

Let the required wave and the pressure $p_0(x)$ possess the same symmetry with respect to a vertical through the crest. Let us align the axis $Oy$ with the axis of symmetry and direct it vertically upward. As the origin of coordinates $O$ we take the point of intersection of the axis $Oy$ with the free surface, and direct the axis $Ox$ to the right. We take the plane of the flow $xOy$ as the plane of the complex variable $z = x + iy$. We introduce the usual notation: $\varphi$ is the velocity potential; $\psi$ is the stream function; $w = \varphi + i\psi$ is the complex velocity potential.

In order to derive from the boundary condition the basic equation of the problem, we first map conformally the region occupied by one wave and constituting an infinite vertical semi-strip bounded above by a wavelike curve, onto the semi-strip $0 \le \varphi \le c\lambda,\ 0 \le \psi \le \infty$ in the $w$-plane, and then this semi-strip onto the interior of the unit circle with center at the origin of the $u = u_1 + iu_2$ plane. It is assumed here that the wavelength $\lambda$ coincides with the period of the function $p_0(x)$. As is well known, the latter mapping is given by the formula

$$w = \frac{\lambda c}{2\pi i} \ln u, \tag{1}$$

where the wave profile is transformed into the circumference of the unit circle with a cut along the radius $\arg u = 0$.

The expression of $z$ in terms of $u$ is determined from the relation

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i}\frac{f(u)}{u}, \qquad f(u) = 1 + \sum_{k=1}^{\infty} a_k u^k. \tag{2}$$

The coefficients $a_k$ are real, since the wave is symmetric with respect to the axis $Oy$.

As usual, introducing the function (1)

$$\omega(u) = \Phi + i\tau = -i \ln f(u), \tag{3}$$

we find from (2) and (3), for $u = e^{i\theta}$ ($\theta$ is the angle of the radius vector with the axis $u_1$), a differential relation; separating in it the real and imaginary parts and integrating, we obtain the parametric equation of the wave profile

$$x = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau(\eta)} \cos\Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau(\eta)} \sin\Phi(\eta)\, d\eta. \tag{4}$$

It follows from formulas (3), (2), and (1) that everywhere in the flow the function $\Phi$ is equal to the angle of the velocity vector $\mathbf{q}$ with the axis $Ox$, and that

$$q = |\mathbf{q}| = c\exp(\tau). \tag{5}$$

From Bernoulli' s integral for the surface, taking into account, by Laplace' s law, the forces of surface tension, after transformations we obtain

$$\frac{d\Phi}{d\theta} = \nu\left[\delta e^{-\tau} - e^{\tau} - \frac{2\pi}{\lambda}\varkappa y e^{-\tau} - p_0^*(x)e^{-\tau}\right], \tag{6}$$

where

$$\nu = \lambda c^2 \rho / 4\pi\mu, \qquad \delta = 2(C\rho - p_0')/\rho c^2, \tag{7}$$

$$\varkappa = g\lambda/\pi c^2, \qquad p_0^*(x) = 2p_0(x)/\rho c^2;$$

$C$ is the constant in Bernoulli' s integral, $g$ is the acceleration of gravity, $\rho$ is the density, and $\mu$ is the capillary constant. In (6), $y$ is determined by the second formula (4). Separating in the right-hand side of (6) the terms linear with respect to $\Phi$ and $\tau$, we obtain

$$\frac{d\Phi}{d\theta} = \nu \left\{ \delta - 1 + (\delta + 1)\tau + \varkappa \int_0^\theta \Phi(\eta)\, d\eta - S(\theta)(1 - \tau) + F[\tau, \Phi, S, \delta] \right\}, \tag{8}$$

where

$$F[\tau, \Phi, S, \delta] = \delta(e^{-\tau} - 1 + \tau) - (e^{\tau} - 1 - \tau) + \varkappa e^{-\tau} \int_0^\theta \left[ e^{-\tau(\eta)} \sin\Phi - \Phi(\eta) \right] d\eta - \varkappa \int_0^\theta \Phi(\eta)\, d\eta + \varkappa e^{-\tau} \int_0^\theta \Phi(\eta)\, d\eta - S($$

Here it is assumed that, up to the constant included in $p_0'$,

$$p_0^*(x) = S(\theta) = \sum_{n=1}^{\infty} \varepsilon^n d_n \cos n\theta, \tag{9}$$

where $\varepsilon$ is a small positive dimensionless parameter, and $d_n$ are given real numbers, the series

$$\sum_{n=1}^{\infty} \varepsilon^n d_n$$

converging in the circle $\varepsilon_0 > 0$.

Let us note that in the original problem $p_0^*(x)$ is a prescribed periodic function of $x$. It can, however, be shown that the solution of the problem under study, under condition (9), corresponds to prescribing the series

$$p_0^*(x) = \sum_{n=1}^{\infty} \varepsilon^n c_n' \cos \frac{2\pi n}{\lambda} x, \qquad c_n' = \sum_{m=0}^{\infty} \varepsilon^m c_{mn}'.$$

In this case, either the coefficients $c'_{0n}$ can be regarded as given and $d_n$ determined from them, or, conversely, the coefficients $c'_{mn}$ $(m = 1, 2, ...)$ are determined through $d_n$.

Equality (8) gives a relation between the functions $\tau(\theta)$ and $\Phi(\theta)$ on the circle $|u| = 1$. For them the known Dini relations hold:

$$\Phi(\theta) = \int_0^{2\pi} K_0(\eta, \theta)\frac{d\tau}{d\eta}\, d\eta, \qquad K_0(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\eta\, \sin n\theta}{n},$$

$$\tau(\theta) = -\int_0^{2\pi} K(\eta, \theta)\frac{d\Phi}{d\eta}\, d\eta, \qquad K(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\eta\, \cos n\theta}{n}. \qquad (10)$$

We transform the linear terms in (8), using formulas (10) and integration by parts. Then we combine the terms with different kernels

$$K(\eta, \theta) \quad \text{and} \quad K_2(\eta, \theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\eta\, \cos n\theta}{n^2}$$

(the first iteration of the kernel $K(\eta, \theta)$) and the same integrand $d\Phi/d\eta$.

In equation (8) the constants $\nu$ and $\chi$ are regarded as given, while $\delta$ is determined from the periodicity condition: $\Phi(\theta+2\pi) = \Phi(\theta)$. Since $S(\theta)$ is given by formula (9), the solution of equation (8), and consequently $\delta$, will depend on $\varepsilon$. Setting in (8)

$$\delta = \delta_0 + \delta'(\varepsilon), \qquad (11)$$

we find that $\delta_0 = 1$ from the periodicity condition as $\varepsilon \to 0$.

After all transformations and taking (11) into account, equation (8) assumes the final form:

$$\zeta(\theta) = \nu\Bigg\{\int_0^{2\pi} K^*(\eta, \theta)\zeta(\eta)\, d\eta + \delta'(\varepsilon) + \delta'(\varepsilon)\int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\, d\eta +$$

$$+\chi\int_0^{2\pi} K_2(\eta, \theta)\zeta(\eta)\, d\eta - S(\theta)\left[1 + \int_0^{2\pi} K(\eta, \theta)\zeta(\eta)\, d\eta\right] +$$

$$+F[\tau, \Phi, S, 1 + \delta'(\varepsilon)]\Bigg\}. \qquad (12)$$

Here $d\Phi/d\theta = \zeta(\theta)$,

$$K^*(\eta,\theta) = \sum_{n=1}^{\infty} \frac{\varphi_n(\eta)\varphi_n(\theta)}{\nu_n}, \qquad \nu_n = \frac{n^2}{2n-\chi}, \qquad \varphi_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}; \tag{13}$$

$\nu_n$ are the eigenvalues, and $\varphi_n(\theta)$ are the eigenfunctions of the kernel $K^*(\eta,\theta)$.

If one considers that, in the expression $F$, the function $\tau(\theta)$ is taken from (10) and

$$\Phi(\theta) = \int_0^{\theta} (d\Phi/d\eta)\, d\eta,$$

then (12) will be a nonlinear integral equation for $\zeta(\theta) = d\Phi/d\theta$.

The periodicity condition for the function $\Phi(\theta)$ gives the relation

$$\delta'(\varepsilon) = -\chi \int_0^{2\pi} K_2(\eta,0)\zeta(\eta)\, d\eta + \frac{1}{2\pi}\int_0^{2\pi} \left\{ S(\theta)\left[1+\int_0^{2\pi} K(\eta,\theta)\zeta(\eta)\, d\eta\right] - F[\tau,\Phi,S,1+\delta'(\varepsilon)] \right\} d\theta. \tag{14}$$

Thus the problem has been reduced to determining the function $\zeta(\theta,\varepsilon)$ and the constant $\delta'(\varepsilon)$ from equations (12) and (14). Here $\tau(\theta,\varepsilon)$ is found from (10), and

$$\Phi(\theta,\varepsilon) = \int_0^{\theta} \zeta(\eta,\varepsilon)\, d\eta.$$

In solving it, two cases have to be considered: in the first case $\nu \neq \nu_n$, and in the second $\nu = \nu_n$.

In the first case, the solution $\xi(\theta,\varepsilon)$ and $\delta'(\varepsilon)$ is constructed in the form of series in integral powers of the parameter $\varepsilon$. In the second case, as an example, we have considered the value $v = v_1$. Here the solution is obtained in the form of series in powers of $\varepsilon^{1/3}$. In both cases, applying the methods of Lyapunov-Schmidt with the use (for $v = v_1$) of the Newton diagram [5], we prove that the indicated series converge absolutely and uniformly for $0 \le \theta \le 2\pi$ and small values $|\varepsilon| < \varepsilon_1 \le \varepsilon_0$, and give the unique solution of the problem that is small relative to $\varepsilon$ and continuous in $\theta$.

With regard to the case $v = v_n$, we further note that, by virtue of (13), $v_n$ depends on the parameter $\chi$ and, as can be shown, for certain $\chi$, called bifurcation values, the eigenvalue $v_n$ will be double. Moreover, it may also be negative. The parameter $v > 0$ according to (7). Here it is assumed that the parameter $\chi$ is chosen so that the eigenvalue $v_1$ is simple and positive.

The wave profile in parametric form $x(\theta, \varepsilon)$ and $y(\theta, \varepsilon)$ is determined from relations (4). Eliminating $\theta$ from the parametric equation, we obtain the profile equation in the form $y = y(x, \varepsilon)$.

We give the wave-profile equations, approximate up to terms of third order, in both cases, putting $k = 2\pi/\lambda$.

In the case $v \neq v_n$:

$$y(x, \varepsilon) = \frac{1}{k}\left\{\varepsilon C_{11}(\cos kx - 1) + \frac{1}{4}\varepsilon^2(C_{11}^2 - C_{22})(1 - \cos 2kx) + \right.$$

$$+\frac{1}{6}\varepsilon^3\left[(6C_{13} + \frac{3}{4}C_{11}C_{22})(\cos kx - 1) + \right.$$

$$\left.\left. + \left(\frac{1}{3}C_{11}^3 - \frac{5}{4}C_{11}C_{22} + \frac{2}{3}C_{33}\right)(\cos 3kx - 1)\right]\right\}, \tag{15}$$

where

$$C_{11} = \frac{vv_1d_1}{v - v_1}, \qquad C_{22} = \left(d_2 + \frac{1}{2}d_1C_{11} + \frac{3}{4}\chi C_{11}^2\right) vv_2/(v - v_2),$$

$$C_{13} = \frac{vv_1}{v_1 - v}\widetilde{C}_{13}, \qquad C_{33} = \frac{vv_3}{v_3 - v}\widetilde{C}_{33}.$$

Here $\widetilde{C}_{13}$ and $\widetilde{C}_{33}$ are linear functions of $C_{11}^3$, $C_{11}^2d_1$, $C_{11}d_2$, $C_{11}C_{22}$, $C_{22}d_1$, and $d_3$.

In the case $v = v_1$, the expression for $y(x, \varepsilon)$ is obtained from (15) if, in the curly bracket, the terms

$$\varepsilon^2C_{12}(\cos kx - 1), \qquad \frac{1}{2}\varepsilon^3(C_{11}C_{12} - \frac{1}{2}C_{23})(1 - \cos 2kx)$$

are added, and everywhere $\varepsilon$ is replaced by $\varepsilon^{1/3}$. The coefficients will have the values:

$$C_{11} = d_1^{1/3}\alpha^{1/3}, \qquad \alpha = \frac{32(v_2 - v_1)}{8(v_2 - v_1) + 9\chi^2v_1v_2},$$

$$C_{22} = \frac{3}{4}\chi C_{11}^2\frac{v_1v_2}{v_1 - v_2}, \qquad C_{12} = -\frac{\chi\left(\frac{1}{2}C_{11}^2 + 5C_{22}\right)}{9\left[1 + \chi\left(1 + 7v_1v_2\chi/8(v_2 - v_1)\right)\right]},$$

$$C_{23} = \chi C_{11}C_{12}\frac{v_1v_2}{v_1 - v_2}, \qquad C_{33} = \frac{v_1v_3}{v_3 - v_1}\widetilde{C}_{33};$$

$\widetilde{C}_{33}$ is a linear function of $C_{11}^3$ and $C_{11}C_{22}$; $C_{13}$ is analogous to $C_{12}$, but with different coefficients.

According to the condition of the problem, the origin of coordinates is placed at the wave crest. Therefore, from an analysis of the principal terms in the formulas for $y(x, \varepsilon)$, and assuming $v_1 < v < v_2$, we conclude that one must take $d_1 > 0$.

We note that $v = v_1$ is precisely the special case that was noted at the beginning of the article. Indeed, for $v = v_1$, from (7) and (13) one obtains the known formula connecting $c$ and $\lambda$ in the indicated special case.

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*Note: Figure translations are in progress. See original paper for figures.*

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