MATHEMATICS

Academician N. N. KRASOVSKII, A. I. SUBBOTIN

1970

SovietRxiv

Abstract

Full Text

MATHEMATICS

Academician N. N. KRASOVSKII, A. I. SUBBOTIN

ON THE STRUCTURE OF DIFFERENTIAL GAMES

Let us consider a differential game of minimax time to encounter. The purpose of the present work, which is related to the investigations (1-11), is to clarify the structure of the game in the class of approximation strategies. Let the system be described by the equation

$$\dot{x} = f^{(1)}(t, x, u) + f^{(2)}(t, x, v), \tag{1}$$

where x is the phase vector; u, v are control vectors subordinate to the first and second players and constrained by the condition

$$u \in \mathcal{U}, \qquad v \in \mathcal{V}, \tag{2}$$

where the sets $\mathcal{U}$ and $\mathcal{V}$ are bounded and closed; $f^{(i)}$ are continuous functions satisfying the Lipschitz condition in x. Denote by the symbol $\mathcal{F}^{(i)}(t, x)$ the convex hull of the set traversed by the vector $f^{(i)}(t, x, w)$, when $w = u$ or $w = v$ ranges over $\mathcal{U}$ or $\mathcal{V}$. We define **strategies** U and V by systems of sets $\mathcal{F}_U(t, x) \subset \mathcal{F}^{(1)}(t, x)$, $\mathcal{F}_V(t, x) \subset \mathcal{F}^{(2)}(t, x)$, calling a **motion** any absolutely continuous vector-function $x[t]$ satisfying the contingent equation $\dot{x}[t] \in \mathcal{F}_U(t, x) + \mathcal{F}_V(t, x)$ for almost all t (the symbol $\mathcal{F}_U + \mathcal{F}_V$ denotes the algebraic sum of sets). Strategies U_T and V_T, determined by the sets $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$, will be called trivial. We define **approximation strategies** U_a and V_a by systems of sets $\mathcal{U}_\Delta(t, x)$ and $\mathcal{V}_\Delta(t, x)$, defined for all possible positions $\{t, x\}$ under any choice of a covering Δ ($\tau_i \le t < \tau_{i+1},\ \tau_0 = t_0,\ \max(\tau_{i+1} - \tau_i) = \delta$) of the half-axis $t \ge t_0$. For each covering Δ, the strategy $U_a(V_a)$, paired with some strategy $V(U)$, determines a motion $x_\Delta[t]$ as an absolutely continuous function which, for almost all t, satisfies the contingent equation

$$\dot{x}_\Delta[t] \in f^{(1)}(t, x_\Delta[t], u_\Delta[\tau_i]) + \mathcal{F}_V(t, x_\Delta[t]), \qquad u_\Delta[\tau_i] \in \mathcal{U}_\Delta(\tau_i, x_\Delta[\tau_i]), \quad \tau_i \le t < \tau_{i+1}, \tag{3}$$

or the contingent equation

$$\dot{x}_\Delta[t] \in \mathcal{F}_U(t, x_\Delta[t]) + f^{(2)}(t, x_\Delta[t], v_\Delta[\tau_i]), \tag{4}$$

$$v_\Delta[\tau_i] \in \mathcal{V}_\Delta(\tau_i, x_\Delta[\tau_i]), \quad \tau_i \leq t < \tau_{i+1},$$

respectively. If, in describing the motion, the strategy of only one player is specified, this means that the strategy of the other player is trivial.

The game problem under consideration, that of encounter with a given closed set $\mathcal{M}$ in the class of strategies U_a and V_a, is formulated as follows. The initial state $x[t_0] = x_0$ is given. Let

$$\gamma^0(U_a) = \sup_{\varepsilon>0} \left(\limsup_{\delta\to 0} \left[\sup_{x_\Delta[t]} \vartheta^\varepsilon_{x_\Delta[t]} \right] \right), \tag{5}$$

where $\vartheta^\varepsilon_{x[t]}$ is the moment when $\rho(x[t], \mathcal{M}) \leq \varepsilon$ for the first time, with $\rho(x, \mathcal{M})$ being the distance from x to $\mathcal{M}$ (recall that in (5) $x_\Delta[t]$ is the motion generated by the strategies U_a and V_T, and, moreover, $x[t_0] = x_0$). A strategy U_a^0, satisfyoaicite: NULL

condition

$$\gamma^0(U_a^0) = \lim_{\varepsilon\to 0} \left(\inf_{\delta>0} \left[\inf_{U_a} \left(\sup_{x_\Delta[t]} \vartheta^\varepsilon_{x_\Delta[t]} \right) \right] \right) \tag{6}$$

will be called **minimax** ($x_\Delta[t]$—the motion generated by the strategies U_a and V_T). Further, let

$$\gamma_0(V_a) = \lim_{\varepsilon\to 0} \left(\liminf_{\delta\to 0} \left[\inf_{x_\Delta[t]} \vartheta^\varepsilon_{x_\Delta[t]} \right] \right) \tag{7}$$

($x_\Delta[t]$—the motion generated by the strategies U_T and V_a). A strategy V_a^0 satisfying the condition

$$\gamma_0(V_a^0) = \sup_{\varepsilon>0} \left(\sup_{\delta>0} \left[\sup_{V_a} \left(\inf_{x_\Delta[t]} \vartheta^\varepsilon_{x_\Delta[t]} \right) \right] \right), \tag{8}$$

will be called **maximin** ($x_\Delta[t]$—the motion generated by the strategies U_T and V_a). The desired saddle point $\{U_a^0, V_a^0\}$ is determined by the equality $\gamma^0(U_a^0) = \gamma_0(V_a^0)$, and the value $\gamma^0(U_a^0)$ will be called the **value of the game**.

Let a system of closed sets $\mathcal{W}(t)$ ($t_0 \leq t \leq \vartheta$) be given. The strategy $U_a^{(e)}$ ($V_a^{(e)}$) extremal to them is defined by the sets $\mathcal{U}_\Delta^{(e)}$ ($\mathcal{V}_\Delta^{(e)}$) as follows. If $x \in \mathcal{W}(t)$, then

$\mathcal{U}_\Delta^{(e)}(t,x) = \mathcal{U}$ ($\mathcal{V}_\Delta^{(e)}(t,x) = \mathcal{V}$). If, however, $x \notin \mathcal{W}(t)$, then $\mathcal{U}_\Delta^{(e)}(t,x)$ ($\mathcal{V}_\Delta^{(e)}$) consists of all vectors $u = u^e \in \mathcal{U}$ ($v = v^e \in \mathcal{V}$) which satisfy the condition

$$s' f^{(1)}(t,x,u^e) = \max_{u \in \mathcal{U}} s' f^{(1)}(t,x,u) \tag{9}$$

or, respectively, the condition

$$s' f^{(2)}(t,x,v^e) = \max_{v \in \mathcal{V}} s' f^{(2)}(t,x,v), \tag{10}$$

where s ranges over the set of all unit vectors directed from the point x to the nearest points to it of $\mathcal{W}(t)$ (the prime superscript denotes transposition). We shall say that the sets $\mathcal{W}(t)$ are strongly u-stable (strongly v-stable) if, whatever $t_* \in [t_0, \vartheta)$, $x_* \in \mathcal{W}(t_*)$, and $\delta \in (0, \vartheta - t_*]$, for any integrable function $v(t) \in \mathcal{V}$ ($u(t) \in \mathcal{U}$), among the motions $x(t)$ ($x(t_*) = x_*$) generated by the strategies U_T and V_v, for which $\mathcal{V}_\Delta(t,x) = f^{(2)}(t,x,v(t))$ (the strategies V_T and U_u, for which $\mathcal{U}_\Delta(t,x) = f^{(1)}(t,x,u(t))$), there is a motion satisfying the condition $x(t_* + \delta) \in \mathcal{W}(t_* + \delta)$. The sets $\mathcal{W}(t)$ will be called stable if, whatever $t_* \in [t_0, \vartheta)$, $x_* \in \mathcal{W}(t_*)$, and $\delta \in (0, \min\{\vartheta - t_*, \xi_0(x_*, \mathcal{M})\}]$ ($\xi < 0$–a sufficiently small constant), for any integrable function $v(t) \in \mathcal{V}$, among the motions $x(t)$ ($x(t_*) = x_*$) generated by the strategies U_T and V_v, there is a motion satisfying the condition $x(t_* + \delta) \in \mathcal{W}(t_* + \delta)$.

Lemma 1. If $x_0 \in \mathcal{W}(t_0)$, the sets $\mathcal{W}(t)$ ($t_0 \le t \le \vartheta$): (1°) are strongly u-stable, or (2°) are strongly v-stable, or (3°) are u-stable, $\mathcal{M} \subset \mathcal{W}(t)$, $\mathcal{W}(\vartheta) = \mathcal{M}$, then the strategy extremal to them, (1°) $U_a^{(e)}$, (2°) $V_a^{(e)}$, (3°) $U_a^{(e)}$, working in pair with the trivial strategy of the other player, ensures in cases (1°) and (2°) the condition

$$\lim_{\delta \to 0} \sup \left(\sup_{x_\Delta[t]} \left[\sup_{t_0 \le t \le \vartheta} \rho(x[t], \mathcal{W}(t)) \right] \right) = 0 \quad (x_\Delta[t_0] = x_0), \tag{11}$$

and in case (3°) the condition

$$\lim_{\delta \to 0} \sup \left(\sup_{x_\Delta[t]} \left[\inf_{t_0 \le t \le \vartheta} \rho(x[t], \mathcal{M}) \right] \right) = 0 \tag{12}$$

($x_\Delta[t_0] = x_0$).

We shall say that, from the position $\{t_*, x_*\}$ ($t_0 \le t_* \le \vartheta$), the set $\mathcal{M}$ is **absorbed** by the time ϑ if

$$\sup_{\delta > 0} \left(\sup_{V_a} \left[\inf_{x_\Delta[t]} \left(\inf_{t_* \le t \le \vartheta} \rho(x_\Delta[t], \mathcal{M}) \right) \right] \right) = 0. \tag{13}$$

(where $x_\Delta[t]$ $(x_\Delta[t_*] = x_*)$ is the motion generated by the strategies V_a and U_τ). Let $\mathcal{W}^p(t, \vartheta)$ be the set of all points x for which $\mathcal{M}$ is absorbed by the time ϑ from the position $\{t, x\}$.

Theorem 1. The sets $\mathcal{W}^p(t, \vartheta)$ $(t_0 \leq t \leq \vartheta)$ are u-stable. Let $x_0 \in \mathcal{W}^p(t_0, \vartheta)$. Then the strategy $U_a^{(e)}$, extremal to $\mathcal{W}^p(t, \vartheta)$, satisfies the condition

$$\gamma^0(U_a^{(e)}) \leq \vartheta. \tag{14}$$

(1) If $\vartheta = \vartheta_0(t_0, x_0)$ is the least of the numbers satisfying the absorption condition (13) from the position (t_0, x_0), then $\gamma^0(U_a^{(e)}) = \vartheta_0$, $U_a^{(e)}$ is a minimax strategy, and ϑ_0 is the value of the game, i.e. $\vartheta_0 = \gamma^0(U_a^{(e)}) = \gamma_0(V_a^{(0)})$.

(2) If equality holds in (14) and $U_a^{(e)}$ is a minimax strategy, then ϑ is the value of the game, and the maximin strategy V_a^0 is constructed on the basis of a sequence of strategies $V_{a,j}^{(e)}$, extremal to suitable sets $\mathcal{W}_v^p(t, \vartheta_j)$ $(t_0 \leq t \leq \vartheta_j < \vartheta)$.

The strongly v-stable sets $\mathcal{W}_v(t_*, \vartheta_j)$ $(\lim \vartheta_j = \vartheta)$ are defined as sets of points x_* satisfying the condition

$$\inf_{\delta>0} \left(\inf_{U_a} \left[\sup_{x_\Delta[t]} \left(\inf_{t_* \leq \vartheta_j} \rho(x_\Delta[t], \mathcal{M}) \right) \right] \right) \geq \varepsilon_j, \tag{15}$$

where $\varepsilon_j > 0$ are sufficiently small numbers, and $x_\Delta[t]$ $(x_\Delta[t_*] = x_*)$ is the motion generated by the strategies U_a and V_τ.

We shall say that from the position $\{t_*, x_*\}$ $(t_0 \leq t_* \leq \vartheta)$ the set $\mathcal{M}$ is absorbed at the time ϑ if

$$\sup_{\delta>0} \left(\sup_{V_a} \left[\inf_{x_\Delta[t]} \rho(x_\Delta[\vartheta], \mathcal{M}) \right] \right) = 0 \tag{16}$$

(where $x_\Delta[t]$ $(x[t_*] = x_*)$ is the motion generated by the strategies U_τ and V_a). Let $\mathcal{W}_*(t, \vartheta)$ be the set of all points x for which $\mathcal{M}$ is absorbed at the time ϑ from the position $\{t, x\}$. We have $\mathcal{W}_*(t, \vartheta) \subset \mathcal{W}^p(t, \vartheta)$. If $\vartheta = \vartheta^0(t_0, x_0)$ is the least of the numbers satisfying the absorption condition (15) from the position $\{t_0, x_0\}$, then $\vartheta_0 \leq \vartheta^0$.

Theorem 2. Let $x_0 \in \mathcal{W}_*(t_0, \vartheta)$. Then the sets $\mathcal{W}_*(t, \vartheta)$ $(t_0 \leq t \leq \vartheta)$ are nonempty and strongly u-stable. The strategy $U_a^{(e)}$, extremal to them, satisfies condition (14). If equality holds in this condition and $U_a^{(e)}$ is a minimax strategy, then ϑ is the value of the game, and the maximin strategy V_a^0 is constructed on the basis of strategies $V_{a,j}^{(e)}$ $(j = 1, 2, ...)$, extremal to the sets $\mathcal{W}_v^p(t, \vartheta_j)$.

A game for which the conditions of Theorem 2 are fulfilled will be called a game with **simple structure**. To find the time ϑ^0 and the sets $\mathcal{W}_*(t, \vartheta^0)$ from condition (16) is practically difficult. In order to seek an effective solution of the problem, condition (16) may be replaced by the weaker condition of programmed absorption: we shall say that from the position $\{t_*, x_*\}$ the set $\mathcal{M}$ is absorbed programmatically at the time ϑ if

$$\sup_{v(\cdot)} \inf_{x[t]} \rho(x[\vartheta], \mathcal{M}) = 0 \tag{17}$$

(where $x[t]$ ($x[t_*] = x_*$) is the motion generated by the strategies U_τ and V_v). Let $\mathcal{W}^*(t, \vartheta)$ be the set of all points x for which $\mathcal{M}$ is absorbed programmatically at the time ϑ from the position $\{t, x\}$. We have $\mathcal{W}_*(t, \vartheta) \subset \mathcal{W}^*(t, \vartheta)$. If $\vartheta = \vartheta_p(t_0, x_0)$ is the least of the numbers satisfying the absorption condition (17) from the position $\{t_0, x_0\}$, then $\vartheta^0 \geq \vartheta_p$.

Theorem 3. Let $x_0 \in \mathcal{W}^*(t_0, \vartheta)$ and let the sets $\mathcal{W}^*(t, \vartheta)$ ($t_0 \leq t \leq \vartheta$) be strongly u-stable. Then the strategy $U_a^{(e)}$, extremal to them, satisfies condition (14). If equality holds in this condition and $U_a^{(e)}$ is a minimax strategy, then ϑ is the value of the game.

A game with a simple structure satisfying the conditions of Theorem 3 will be called a game with the **simplest structure**. An example of such a game is provided by the pursuit problem in the case of linear objects of the same type ([9]) under the convex encounter condition $(y - z) \in \mathcal{M}$. In connection with the preceding, effective conditions are of interest which ensure strong u-stability of the sets $\mathcal{W}^*(t, \vartheta)$. For the linear system (1), under convex constraints (2) and for a convex set $\mathcal{M}$, these conditions are as follows ([10,11]): the sets $\mathcal{W}^*(t, \vartheta)$ are described by a certain inequality $\chi(t, \vartheta, l) - l'x \geq 0$, where χ is expressed in a known way through the support functions of the set $\mathcal{M}$ and of the attainability regions of the motion x with respect to u and v; if for every $\varepsilon > 0$ the minimum $\min |\chi(t, \vartheta, l) + \varepsilon - l'x| = 0$ ($\|l\| = 1$) is attained at a single vector l, then the sets $\mathcal{W}^*(t, \vartheta)$ are strongly u-stable. Condition (13) can also be replaced by an analogous condition of programmed absorption by the moment ϑ. A statement analogous to Theorem 1 will be valid, but the u-stability of the corresponding sets $\mathcal{W}^0(t, \vartheta)$ must now be stipulated, and for checking it it is difficult to indicate as effective a criterion as in the case of the sets $\mathcal{W}^*(t, \vartheta)$. Finally, let us note the following. If in condition (9) (or (10)) the vector s is unique, then it is possible to construct not only approximation strategies $U_a^{(\varepsilon)}$ (or $V_a^{(\varepsilon)}$), but also strategies $U^{(\varepsilon)}$ (or sometimes $V^{(\varepsilon)}$) possessing analogous properties, operating within the framework of differential equations in contingencies (see, for example, ([9,10])) and already ensuring, by themselves, the limiting relations corresponding to the minimax (or maximin) conditions.

Sverdlovsk Branch of the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR

Received 24 X 1969

REFERENCES

[1] W. H. Fleming, J. Math. Anal. and Appl., **3**, 102 (1961).
[2] C. R. Nardzewski, Adv. in Game Theory, Ann. of Math. Studies, 1964, p. 113.
[3] R. Isaacs, *Differential Games*, Moscow, 1967.
[4] L. S. Pontryagin, DAN, **175**, No. 4, 764 (1967).
[5] B. N. Pshenichnyi, DAN, **184**, No. 2, 285 (1969).
[6] M. S. Nikol' skii, Vestn. Mosk. Univ., Ser. Math. and Mech., No. 3, 65 (1969).
[7] R. Varaiya, J. Lin, SIAM J. Control, **7**, No. 1, 141 (1969).
[8] N. N. Petrov, DAN, **190**, No. 6 (1970).
[9] N. N. Krasovskii, DAN, **181**, No. 5, 1062 (1968).
[10] N. N. Krasovskii, Izv. AN SSSR, Tekhn. Kibernetika, No. 5, 3 (1969).
[11] N. N. Krasovskii, A. I. Subbotin, PMM, **33**, No. 4, 698 (1969).

Note: Figure translations are in progress. See original paper for figures.