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Abstract

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ON THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR THE HEAT EQUATION

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The purpose of this note is to establish necessary and sufficient conditions under which the solution of the Cauchy problem for the heat equation stabilizes as $t \to \infty$. Let $u(x,t)$, $x = (x_1, \dots, x_n)$, be a solution of the equation

$$u_t = \Delta u, \qquad x \in R_n = (-\infty < x_i < \infty,\ i = 1, \dots, n), \tag{1}$$

satisfying the initial condition

$$u\big|_{t=0} = f(x). \tag{2}$$

We shall assume that $f(x) \in C(R_n) \cap T^2(R_n)$, i.e. f is continuous and satisfies the condition: for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $|f(x)| \leq C(\varepsilon) e^{\varepsilon |x|^2}$ for all $x \in R_n$. It is known (^{1}) that for $f \in T^2(R_n) \cap C(R_n)$ the solution $u(x,t)$ of problem (1)–(2) exists for all $t \in (0, \infty)$, is unique, and is represented in the form of the Poisson integral

$$u(x,t) = \frac{1}{(2\sqrt{\pi t})^n} \int_{R_n} f(\xi) e^{-|x-\xi|^2/4t}\, d\xi \equiv (\mathfrak{B}_t^{(2)} f)(x). \tag{3}$$

By stabilization of the solution $u(x,t)$ as $t \to \infty$ we shall, in different places of this note, mean I or II.

I. The existence of a limit uniform with respect to $x \in R_n$,

$$\lim_{t \to \infty} u(x,t) = A(x). \tag{4}$$

II. The existence of the limit (4), uniform on every compact set $K \subset R_n$.

Together with these two types of stabilization we shall also automatically consider stabilization of the following kind:

III. The limit (4) exists for every $x \in R_n$, since stabilization in the sense III follows from stabilization in the sense II.

A criterion for pointwise stabilization of the solution of problem (1)–(2) in the case of bounded $f(x)$ is due to N. Wiener (^{2}) (see also the works (^{3,4})) and consists in the fact that a necessary and sufficient condition for stabilization, in the sense III, of the solution of problem (1), (2) is the existence, as $R \to \infty$, at each point $x \in R_n$, of the limit of the expression

$$(S_R^1 f)(x) = \frac{n}{\omega_n R^n} \int_{|x-y| \leq R} f(y)\, dy, \tag{5}$$

$\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, equal to $A(x)$. (We note that in this case $A(x) \equiv \text{const.}$) Recently, in (^{5}), the same result was established, but already for the case of semibounded $f(x)$ (either $f(x) \geq M$, or $f(x) \leq M$ for all $x \in R_n$, for some constant M). In this case $A(x)$ is also identically equal to a constant.

Before formulating the result of the article, it is convenient to introduce some concepts. Let

$$(F_\rho f)(x) = \frac{n}{\omega_n} \int_{|x-y|=\rho} f(y)\, dy = \frac{n\rho^{n-1}}{\omega_n} \int_{|\omega|=1} f(x+\rho\omega)\, d\omega \tag{6}$$

(i.e., on $C(R_n)$ an operator F_ρ is defined which, obviously, maps $C(R_n)$ into $C(R_n)$). In terms of the operator F_ρ, the spherical mean (5) can be written as follows:

$$(S_R^1 f)(x) = \frac{1}{R^n} \int_0^R (F_\rho f)(x)\, d\rho.$$

For the function $(F_\rho f)(x)$ one can define Cesàro means with respect to ρ of arbitrary order $\alpha > 0$. With their aid we construct the operators

$$(S_R^\alpha f)(x) = \frac{1}{nB(\alpha, n) R^{n+\alpha-1}} \int_0^R (R-\rho)^{\alpha-1} (F_\rho f)(x)\, d\rho \tag{7}$$

for all $\alpha > 0$, $R > 0$. Note that the domain of definition of the operators S_R^α is all of $C(R_n)$, without any restrictions on the growth of $f(x)$. We shall also introduce the scale of operators

$$(B_R^\sigma f)(x) = \frac{\sigma}{\Gamma(n/\sigma)(4R)^{n/\sigma}} \int_0^\infty e^{-\rho^\sigma/4R} (F_\rho f)(x)\, d\rho; \tag{8}$$

$R > 0$, $\sigma > 0$. The operators B_R^σ are no longer defined on all functions in $C(R_n)$, but only on $C(R_n) \cap T^\sigma(R_n)$, where the set $T^\sigma(R_n)$ consists of those functions which have the following property: for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that $|f(x)| \leq C(\varepsilon)e^{\varepsilon|x|^\sigma}$. In particular, for $\sigma = 2$, B_R^σ coincides with the operator B_R in (3).

Theorem 1. *In order that (4) hold, where the limit is understood in sense I (i.e., uniformly in $x \in R_n$), it is necessary and sufficient that there exist, uniformly in $x \in R_n$,*

$$\lim_{R\to\infty} (S_R^1 f)(x) = A(x).$$

The following example convinces us that, for convergence in sense II (i.e., uniformly in x on any compact set $K \subset R_n$), Theorem 1 may fail to hold. Let $n = 1$, and let $f(x) = x \sin x\alpha$ for some real α. Then the solution of problem (1)–(2) has the form

$$u(x,t) = e^{-\alpha^2 t}(x \sin \alpha x + 2\alpha t \cos \alpha x).$$

$u(x,t) \to 0$ $(t \to \infty)$ uniformly for $|x| \leq a$ for any $a > 0$, but not uniformly in all $x \in R_1$. At the same time

$$(S_R^1 f)(x) = -\frac{\cos \alpha R \cos \alpha x}{\alpha} + \varphi(x,R),$$

where $|\varphi(x,R)| \leq c|x|/R$ with some constant $c > 0$, i.e. $(S_R^1 f)(x)$ has no limit at any point except $x = \pi/2\alpha + k\pi/\alpha$, $k = 0, \pm 1, \ldots$. However, for example, the following holds.

Theorem 2. *In order that the limit (4) exist in sense II, i.e., uniformly on any compact set K in x, in the case when $|f(x)| \leq c(|x|^s + 1)$ for some $s \geq 0$, $c > 0$, it is necessary and sufficient that, uniformly in $x \in K$ for any compact $K \subset R_n$, there exist the limit $(S_R^\alpha f)(x)$ as $R \to \infty$ for $\alpha > s$. Analogously, in the case under consideration, a necessary and sufficient condition is also the existence, uniformly in $x \in K$ for any $K \subset R_n$, of the limit $(B_R^\sigma f)(x)$ as $R \to \infty$ for any $\sigma > 0$.*

The proofs of Theorems 1 and 2 are carried out in the same way and are based on a number of auxiliary assertions. Denote by $\mathfrak{R}^\alpha$ the linear space of functions $f \in C(R_n)$ (without any growth restriction) for which there exists, in the sense of II, the limit

$$\lim_{R\to\infty} (S_R^\alpha f)(x) = A(x) \tag{9}$$

($\mathfrak{R}^\alpha$ is the domain of definition of the operator S_∞^α, defined in the corresponding way), and by $\mathfrak{R}_0^\alpha$ its subspace for which, for $f \in \mathfrak{R}^\alpha$, $A(x) \equiv 0$. Obviously, $\mathfrak{R}_0^\alpha$ is a nonempty subspace, since any finite function $f \in \mathfrak{R}_0^\alpha$.

Lemma 1. For $\alpha > 0$ and convergence understood in the sense of II (uniform on every compact $K \subset R_n$), $\mathfrak{R}^\alpha / \mathfrak{R}_0^\alpha = G$, where G is the linear space of all possible harmonic functions in R_n. In other words, in order that $f \in \mathfrak{R}^\alpha$, it is necessary and sufficient that there exist (uniquely) such functions $f_0 \in \mathfrak{R}_0^\alpha$ and $g \in G$, i.e. $\Delta g = 0$, that

$$f(x) = f_0(x) + g(x).$$

From this lemma, in particular, it follows that, if the limit (9) exists in the sense of II, then the function $A(x)$ is necessarily a harmonic function. If f is bounded or semibounded and $f \in \mathfrak{R}^\alpha$, $\alpha > 0$, then $A(x)$ obviously has the same property (and with the same constants), and then, by Liouville’ s theorem for harmonic functions, $A(x) \equiv \text{const}$.

Similarly, if we denote by $\mathfrak{B}^\sigma$, $\sigma > 0$, the linear space consisting of those $f(x) \in C(R_n) \cap T^\sigma(R_n)$ for which there exists, in the sense of II,

$$\lim_{R \to \infty} (B_R^\sigma f)(x) = A(x). \tag{10}$$

By $\mathfrak{B}_0^\sigma$ we denote the subspace of $\mathfrak{B}^\sigma$ for whose elements $A(x)$ in (10) is identically zero. (It is clear that the set of all finite functions from $C(R_n)$ is contained in $\mathfrak{B}_0^\sigma$, i.e. $\mathfrak{B}_0^\sigma$ is nonempty.)

Lemma 2. For $\sigma > 0$ and convergence understood in the sense of II (uniform on every compact $K \subset R_n$), $\mathfrak{B}^\sigma / \mathfrak{B}_0^\sigma = G$, i.e. in order that $f \in \mathfrak{B}^\sigma$, it is necessary and sufficient that there exist (uniquely) such functions $f_0 \in \mathfrak{B}_0^\sigma$ and $g \in G$ that

$$f = f_0 + g.$$

Lemmas 1 and 2 are consequences of Lemma 3.

Lemma 3. Let the function $K(\alpha)$ be a function of one variable $\alpha \in (0, \infty)$ such that $\alpha^{n-1} K(\alpha) \in L_1(0, \infty)$ for some integer $n > 0$, and

1) $K(\alpha) \geq 0$ for $\alpha \in (0, \infty)$,
2)
$$\int_0^\infty \alpha^{n-1} K(\alpha)\, d\alpha = 1.$$

Then, if for all $x \in R_n$ and $R > 0$ the integral

$$(K_R f)(x) = \frac{1}{nR^n} \int_0^\infty K\left(\frac{\rho}{R}\right) (F_\rho f)(x)\, d\rho$$

exists and there exists

$$\lim_{R\to\infty}(K_R f)(x) = A(x),$$

then $A(x)$ is a harmonic function (the function $(F_\rho f)(x)$ is defined from $f(x)$ by means of equality (6)).

Since the existence of the limit (4) in the sense of II (and, all the more, in the sense of I) is membership of $f(x)$ in the space $\mathfrak{B}^{(2)}$ (understood in the sense of the corresponding convergence), Theorem 1 is a consequence of Lemmas 1, 2, and 4.

Lemma 4. If convergence is understood as uniform in R_n, then $\mathfrak{P}_0^{(2)} = \mathfrak{N}_0^1$.

Theorem 2 is an analogous consequence of Lemmas 1, 2, and 5.

Lemma 5. If convergence is understood as uniform on every compact set $K \subset R_n$, then $\mathfrak{P}_0^{(2)} = \mathfrak{N}_0^\alpha$ for any $\alpha > s$, provided only that $|f(x)| \leqslant \text{const}\,(|x|^{\alpha'} + 1)$ for $\alpha' < \alpha$.

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Note: Figure translations are in progress. See original paper for figures.