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Abstract

Full Text

Mathematics

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SOLUTION OF AN ANTICANONICAL EQUATION WITH PERIODIC COEFFICIENTS

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In the complex Hilbert space W the equation

$$J\,dx/dt = H(t)x \qquad (H(t+\tau) = H(t)) \tag{1}$$

is studied, with an operator $H(t)$ depending periodically on t. Here J is a bounded (together with its inverse) symmetric operator. The principal part (in the sense explained below) of the operator $H(t)$ is positive definite.

Some problems of acoustic and electromagnetic periodic waveguides lead to the abstract equation

$$d^2u/dt^2 = S(t)u, \tag{2}$$

in which the principal part of the operator $S(t)$ is positive definite ([1–3]). Equation (2), in turn, by the substitution $x = (u, du/dt)$, is reduced to equation (1).

In paper ([1]), for a specific form of equation (2), an approach was first indicated which brings the theory of periodic waveguides closer to the existing theory of differential equations with periodic coefficients. In papers ([1–3]), boundary-value problems for equation (2) were studied and some assertions concerning the multipliers of this equation were formulated.

The principal part of the operator $S(t)$ in ([1–3]) did not depend on t. Even in this case, apparently, the question of finding all solutions of equation (2) had not been solved in the literature.

It is known ([2,3]) that, for a constant operator $S(t) \equiv S$, the Cauchy problem for equation (2) (and, consequently, for the corresponding equation (1)) is ill-posed. However, the set of Cauchy data can be decomposed into the direct sum of two subspaces such that the Cauchy problem is well-posed on one subspace to the right and on the other to the left. In this article an analogous result (along with others) is established for equation (1).

Below it is assumed that the operator $H(t)$ is decomposed into the sum $H(t) = H_0(t) + \omega H_1(t)$ of two τ-periodic operators. Here ω is a real parameter, and the operators $H_0(t)$ and $H_1(t)$ satisfy the following conditions:

I. The operator $H_0(t)$ is positive definite,

$$H_0^*(t) = H_0(t) \geq \gamma I$$

($\gamma = \text{const} > 0$), and the operator $H_0^{-1}(t)$ is strongly continuous.

II. The domain of definition of the positive square root $H_0^{1/2}(t)$ is constant, and for any t, s the estimate

$$\left\| H_0^{1/2}(t) H_0^{-1/2}(s) \right\| \leq 1 + \text{const}\, |t - s|$$

holds.

III. The operator $H_0^{-\alpha}(0) H_1(t) H_0^{-\alpha}(0)$ $(0 \leq \alpha < 1/2)$ is defined for all t on a dense set, is strongly differentiable on this set for almost all t, and

$$\| H_0^{-\alpha}(0) H_1(t) H_0^{-\alpha}(0) \| + \left\| \frac{d}{dt} H_0^{-\alpha}(0) H_1(t) H_0^{-\alpha}(0) \right\| \leq \text{const}.$$

1°. Let $L_2([0, \tau], W) = L_2$ be the Hilbert space of square-summable in the Bochner sense [(4)] functions with values in W. The scalar product and norm in L_2 are introduced by the formulas

$$[x, y] = \int_0^\tau (x(t), y(t))\, dt, \qquad |x| = \left(\int_0^\tau \|x(t)\|^2\, dt \right)^{1/2},$$

where $(\cdot, \cdot)$ and $\| \cdot \|$ are the scalar product and norm in W.

With equation (1) there is naturally associated the operator

$$A = A(\omega) = \frac{d}{dt} |F| - G \mathcal{H}(t), \tag{3}$$

acting in the space L_2 and defined on the set $\{x(t) : x(t) \in L_2,\ |F| x(t)$ is a weakly absolutely continuous function, $\frac{d}{dt} |F| x(t) \in L_2,\ x(\tau) = x(0)\}$.

In formula (3) the operators $|F|, G, \mathcal{H}(t) : W \to W$ are defined as follows:

$$F = H_0^{-1/2}(0) J H_0^{-1/2}(0), \qquad |F| = (F^* F)^{1/2}, \qquad G = F^{-1} |F|,$$

$$\mathcal{H}(t) = [H_0^{1/2}(t) H_0^{-1/2}(0)]^* [H_0^{1/2}(t) H_0^{-1/2}(0)] + \omega H_0^{-1/2}(0) H_1(t) H_0^{-1/2}(0).$$

The operator F is bounded symmetric and has an unbounded inverse F^{-1}, $|F|$ is the modulus of the operator F, and $G = G^{-1}$ is a bounded symmetric operator. The operator $\mathcal{H}(t)$ is bounded and defined on a dense set. Below, by $\mathcal{H}(t)$ we mean its closure. From conditions I–III it follows that for almost all t the operator $\mathcal{H}(t)$ is strongly differentiable and its derivative is a uniformly in t bounded operator.

It is easy to see that A is a closed operator in L_2. By the operator A, an operator $B = B(\omega)$ is defined in the space L_2:

$$B \equiv |F|^{-1/2} A |F|^{-1/2}, \tag{4}$$

where $|F|^{-1/2}$ is the positive square root of the operator $|F|^{-1}$.

For the operators (3) and (4) the following assertions hold.

Theorem 1. *For any ω the operator $B = B(\omega)$ is defined on a dense subset of L_2 and is closed. With the exception of certain isolated values of ω, the spectrum $\sigma(B(\omega))$ of the operator $B(\omega)$ consists of isolated eigenvalues of finite multiplicity, to which finite-dimensional root subspaces correspond.*

In what follows, all considerations are carried out for those values of the parameter ω for which, according to Theorem 1, the operator $B(\omega)$ has a discrete spectrum, and only for these values of ω are all assertions formulated below valid. We note that for $\omega = 0$ the operator $B(\omega)$ has a discrete spectrum.

Theorem 2. *In order that the operator $A - \xi|F|$ be boundedly invertible, it is necessary and sufficient that ξ be a regular point of the operator B.*

For the validity of Theorems 1 and 2 there is no need to require differentiability or even continuity of the operator $\mathcal{H}(t)$. The smoothness of $\mathcal{H}(t)$ will be needed below.

The operator A (in other terms) was introduced for equation (2) in [1–3]. Nevertheless, the spectrum of the linear pencil $A - \xi|F|$ was not systematically studied. We note that linear pencils were studied in [5]. However, the general theory does not make it possible to establish the discrete character of the spectrum of the pencil $A - \xi|F|$ without using the concrete special features of the operator A.

2°. Define in L_2 two more operators: A^0 and $B^0 \equiv |F|^{-1/2} A^0 |F|^{-1/2}$. The operator A^0 is defined as the special case of the operator A when $H(t) \equiv H_0(0)$. The operators A^0 and B^0 possess all the properties of the operators A and B indicated above in Theorems 1 and 2.

Let a be an arbitrary real number. On the line $a + i\lambda$ $(-\infty < \lambda < +\infty)$ there may lie points of the spectra of the operators B and B^0. We agree to shift the line to the right by a sufficiently small number $\varepsilon > 0$ from the spectra of the operators B and B^0, so as not to cross other spectral points lying to the right of the line $a + i\lambda$ $(-\infty < \lambda < +\infty)$. Below, in formulas (5), integration will be taken along the shifted line $\varepsilon + a + i\lambda$ $(-\infty < \lambda < +\infty)$. It is easy to see that the indicated shift of the line is possible, since the spectrum $\sigma(B)$ has no accumulation points at a finite distance and possesses the property of periodicity: if $\xi \in \sigma(B)$, then $\xi + i \cdot 2\pi k/\tau \in \sigma(B)$ $(k = \pm 1, \pm 2, \ldots)$.

Define in the space W the operators $Y_{\pm,a}(t)$, $P_{\pm,a}(t)$ by the formulas

$$Y_{\pm,a}(t) = Y^0_{\pm a}(t) \pm \frac{1}{2\pi i} \int_{a+\varepsilon+i\infty}^{a+\varepsilon-i\infty} e^{-\xi t} T(\xi)\, d\xi \quad (t \geq 0,\ t \leq 0),$$

$$P_{\pm,a}(t) = P^0_{\pm,a} \pm \frac{1}{2\pi i} \int_{a+\varepsilon+i\infty}^{a+\varepsilon-i\infty} T(\xi)\, d\xi. \tag{5}$$

Here

$$P^0_{+,a} = \int_{-\infty}^{a-\varepsilon} d\bar{E}_\lambda \quad (\bar{E}_\lambda \text{ is the spectral family of the operator } F^{-1}),$$

$$P^0_{-,a} = I - P^0_{+,a}, \qquad Y^0_{\pm,a}(t) = e^{F^{-1}t} P^0_{\pm,a}, \qquad T(\xi) = (\xi I - B)^{-1} - (\xi I - B^0)^{-1}.$$

Formulas (5) require clarification. The integrals in (5) converge strongly in the space W, i.e., converge on every element $\varphi \in W$; moreover, in (5) the function $T(\xi)\varphi$, specified on the interval $t \in [0, \tau]$, should be periodically extended, with period τ, to the entire real axis $-\infty < t < +\infty$.

Theorem 3. *Formulas (5) determine bounded operators in W, depending continuously on t in the strong operator topology.*

We note that the operators (5) do not depend on ε, if, when ε is varied, the line $a + \varepsilon + i\lambda$ $(-\infty < \lambda < +\infty)$ does not cross spectral points of the operators B and B^0.

The proof of the convergence of the integrals (5) is nontrivial and uses the smoothness properties of the operators $H_0(t)$, $H_1(t)$.

Instead of equation (1), consider the equation

$$\frac{d}{dt} y - G|F|^{-1/2} \mathcal{H}(t) |F|^{-1/2} y = 0. \tag{6}$$

The solutions of equations (1) and (6) are connected by the formal relation

$$y = |F|^{1/2} H_0^{1/2}(0) x.$$

Formulas (5) solve equation (6) in the sense formulated below.

Theorem 4. *The operators (5) satisfy the relations*

$$P^2_{\pm,a}(t) = P_{\pm,a}(t), \qquad P_{\pm,a}(t)P_{\mp,a}(t) = 0, \qquad P_{+,a}(t) + P_{-,a}(t) = I,$$

$$P_{\pm,a}(t)Y_{\pm,a}(t) = Y_{\pm,a}(t)P_{\pm,a}(0) = Y_{\pm,a}(t), \qquad Y_{\pm,a}(0) = P_{\pm,a}(0).$$

The space of initial data decomposes into the direct sum of subspaces

$$W = W_{+,a} + W_{-,a}, \qquad W_{\pm,a} = P_{\pm,a}(0)W.$$

For any $y_0 \in W_{\pm,a}$, the function $y_{\pm,a}(t) = Y_{\pm,a}(t)y_0$ is a solution of equation (6) on the interval $[0, \pm\infty)$. As $t \to \pm\infty$, the solutions $y_{\pm,a}(t)$ respectively have the properties $e^{at}y_{+,a}(t) \to 0$, $e^{-(a+\delta)t}y_{-,a}(t) \to 0$ for sufficiently small $\delta > 0$, uniformly with respect to y_0.

Here, by a solution of equation (6) we mean a strongly continuous function which satisfies (6) in the generalized sense. Without giving

For an exact determination of the solution, let us note that if on the interval $[a, b]$ $y(t)$ is a solution satisfying the condition $y(a) = y_0$, then it is unique.

3°. On the invariant subspace $W_{-,\alpha}$ the operator $Y_{-,\alpha}(-t)$ is invertible. Therefore one can define the monodromy operator of equation (6) by the formula

$$Y(\tau) = Y_{+,\alpha}(\tau) + [Y_{-,\alpha}(-\tau)]^{-1}.$$

As was to be expected, the definition of $Y(\tau)$ does not depend on the choice of the number α.

Theorem 5. *The definition of the monodromy operator $Y(\tau)$ of equation (6) does not depend on α. The operator $Y(\tau)$ has spectrum*

$$\sigma(Y(\tau)) = \{\rho : \rho = e^{-\xi\tau},\ \xi \in \sigma(B)\}.$$

Thus, the spectrum of $Y(\tau)$ consists of isolated eigenvalues of finite multiplicity with two (possible) points of accumulation $\rho = 0, \rho = \infty$.

It is also easy to establish the connection and identical structure of the root subspaces of the operators $Y(\tau)$ and B, respectively, for the points of the spectra $\rho = e^{-\xi\tau}$ and ξ of the operators $Y(\tau)$ and B.

4°. Suppose that the following two further conditions are satisfied:

IV. The operator $|F|^{-1/2}\mathcal{H}(t)|F|^{1/2}$ is bounded, strongly differentiable with respect to t for almost all t, and

$$\left\||F|^{-1/2}\mathcal{H}(t)|F|^{1/2}\right\|+\left\|\frac{d}{dt}|F|^{-1/2}\mathcal{H}(t)|F|^{1/2}\right\|\leq \text{const}.$$

V. Let

$$\mathcal{H}_\lambda(t)=\mathcal{H}(t)-(E_\lambda-E_{-\lambda})\mathcal{H}(t)(E_\lambda-E_{-\lambda})+(E_\lambda-E_{-\lambda}),$$

where E_λ is the resolution of the identity of the operator F^{-1}. For sufficiently large λ the positive definiteness of the operator

$$\operatorname{Re}(|F|^{-1/2}\mathcal{H}_\lambda(t)|F|^{1/2}:\operatorname{Re}(|F|^{-1/2}\mathcal{H}_\lambda(t)|F|^{1/2}))\geq cI,\qquad c=\text{const}>0$$

is required.

Here the independence of conditions I-V is not discussed.

It can be shown that the spectrum of the linear pencil $\tilde{A}-\xi|F|$, where the operator $\tilde{A}$ is defined by the formula

$$\tilde{A}=\frac{d}{dt}|F|-G|F|^{-1/2}\mathcal{H}(t)|F|^{1/2}$$

on the set $D(A)$, coincides in the space L_2 with the spectrum of the pencil $A-\xi|F|$.

Using the last assertion, it is established that there exist in the space W bounded and strongly continuous in t operators

$$\begin{aligned}Q_{\pm,\alpha}(t)&=|F|^{-1/2}P_{\pm,\alpha}(t)|F|^{1/2}\quad(-\infty<t<+\infty),\\ Z_{\pm,\alpha}(t)&=|F|^{-1/2}Y_{\pm,\alpha}(t)|F|^{1/2}\quad(t\geq 0,\ t\leq 0).\end{aligned}$$

With the aid of the operator solutions $Z_{\pm,\alpha}(t)$, solutions of equation (1) are constructed. Let $x_0\in D(H_0^{1/2}(0))$; then the function

$$x(t)=H_0^{-1/2}(0)Z_{\pm,\alpha}(t)H_0^{1/2}(0)x_0$$

is a generalized solution of equation (1) on the interval $(0,\pm\infty)$.

One can define the monodromy operator of the equation

$$\frac{d}{dt}Fz=\mathcal{H}(t)z$$

by the formula $Z(\tau) = |F|^{-1/2}Y(\tau)|F|^{1/2}$ and show that the spectra of the operators $Z(\tau)$ and $Y(\tau)$ coincide.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.