ON INTEGRAL CONSERVATION LAWS

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N. P. KONOPLEVA

ON INTEGRAL CONSERVATION LAWS IN GENERAL RELATIVITY

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In this note it is shown that the integral conservation laws of the "moment," "energy," and "momentum" of a system of bodies in a curved V_4 can be obtained from a single covariant differential conservation law for the symmetric energy-momentum tensor of this system, under the assumption that space-time V_4 admits a certain group of motions.*

As is known ([1],[2]), in a Riemannian space the equality of vectors assigned to different points of it has no absolute meaning, since these vectors transform differently upon transition to a new coordinate system. For this reason integration in a Riemannian space is defined only for scalars. Since the volume element is a completely antisymmetric quantity, the scalar under the integral sign is formed by contracting some antisymmetric tensor with the volume element. Symmetric tensors do not yield a scalar under such contraction. Therefore, to integrate them one must use artificial devices; for example, first contract a symmetric tensor with certain additionally introduced quantities to obtain a vector, and then integrate this vector. It is precisely in this way, by considering the contraction of the energy-momentum tensor T^{ik} with the Killing vector ξ^i, which defines the group of motions of space-time V_4, that one can formulate integral conservation laws in general relativity ([4],[5]). It will be shown below that in this case, instead of the set of integral conservation laws of moment, energy, and momentum that hold in flat V_4, there arises one general conservation law, obtained from the covariant differential conservation law for the symmetric energy-momentum tensor T^{ik},

$$T^{ik}{}_{;k} = 0. \tag{1}$$

The character and the number of quantities conserved integrally are then determined by the structure of the group of motions, i.e., by the form of the Killing vector. The narrower the group of motions of V_4, the fewer components of "energy," "momentum," and "moment" that are conserved integrally we can define.

Let us consider Green' s formula in a Riemannian space ([4])

$$\int J^m{}_{;m} d_4 v = \oint J^m \varepsilon(N) N_m d_3 v, \tag{2}$$

where N_m is the normal vector; $\varepsilon(N)$ is the indicator of the vector N_m; $d_4 v$ and $d_3 v$ are invariant volume elements.

* By impulses and moments here are meant quantities which, in a particular coordinate system, have a form analogous to the corresponding quantities in the Cartesian coordinate system in flat V_4. In a Riemannian space this terminology is somewhat conventional, since the division of conserved quantities into impulses and moments becomes dependent on the coordinate system.

For a symmetric tensor of rank two T^{ik}, a formula analogous to formula (2) can be obtained with the aid of an additional vector λ_i ([4,5])

$$\int (T^{ik}\lambda_k)_{;i}\, d_4 v = \int T^{ik}{}_{;k}\lambda_k\, d_4 v + \int T^{ik}\lambda_{i;k}\, d_4 v. \tag{3}$$

It is clear from formula (3) that an integral conservation law for a symmetric tensor has meaning only along certain directions, namely along the trajectories of the vector field λ_k. If a Riemannian space does not admit the existence of any vector field (which is possible in the general case), then no integral conservation laws can be found in this space. True, for a Riemannian space having a metric with Minkowski signature, this restriction is inessential, since the condition for the existence of such a metric coincides with the condition for the existence on the manifold of a continuous field of directions ([6]). Thus, in spaces of general relativity there always exists some vector field and, consequently, one can always construct integral conserved quantities (at least one). The character of the vector field λ_k determines the properties of the conserved integral quantity. Although we integrate the differential conservation law for the energy-momentum tensor, the integral quantity may have quite a different meaning. Let us note, for example, that if the Lagrangian does not contain second derivatives of the field variables, then any conservation current generated by a group G_r can be reduced to the form:

$$J^\mu_a = T^\mu_\nu \xi^\nu_a - \frac{\partial L}{\partial \psi_{,\mu}} I\psi_a,$$

where $\delta x^\nu = \xi^\nu_a \varepsilon^a$, $\delta\psi = I\psi\varepsilon^a$, ε^a are the parameters of G_r. Thus, for scalar representations of G_r ($\delta\psi = 0$), the properties of J^μ_a are determined by the properties of T^μ_ν and the vectors ξ^ν_a.

Let λ_k determine a group of motions of space-time, i.e., let it be a Killing vector: $\lambda_k = \xi_k$, with $\xi_{(k;i)} = 0$. Then relation (3) takes the form, reminiscent of formula (2):

$$\int T^{ik}{}_{;i}\xi_k \, d_4v = \oint T^{ik}\xi_k\varepsilon(N)N_i \, d_3v. \tag{4}$$

In the special case of flat space, ξ^k has the form: 1) for translations, $\xi^k_l = \delta^k_l$; 2) for rotations, $\xi^k_{(in)} = L_{(in)}{}^k{}_l x^l$, where $L_{(in)}{}^k{}_l = \delta^k_n g_{il} - \delta^k_i g_{nl}$ is the matrix of Lorentz rotations.

From formula (4), in the first case we obtain

$$\int T^{ik}{}_{;i} \, d_4v = \oint T^{ik}\varepsilon(N)N_i \, d_3v,$$

whence, taking (1) into account, upon integrating over infinitely distant surfaces where T^{ik} vanishes, we obtain the integral conservation law for the 4-momentum P^i, where

$$P^i = \int T^{i0}\varepsilon(N)N_0 \, d_3v_3.$$

In the case of rotations of flat space, formula (4) gives

$$\int T^{ik}{}_{;i}L_{(pq)kn}x^n \, d_4v = \int (T^{ik}L_{(pq)kn}x^n)_{;i} \, d_4v -$$

$$- \int T^{ik}L_{(pq)ki} \, d_4v = \oint T^{ik}L_{(pq)kn}x^n\varepsilon(N)N_i \, d_3v.$$

If the tensor T^{ik} is symmetric, from this we obtain the usual conservation law for the angular momentum of the system

$$\int M_{(pq)}{}^i{}_{;i} \, d_4v = \oint M_{(pq)}{}^i\varepsilon(N)N_i \, d_3v,$$

where

$$M^i_{(pq)} = T^i_p x_q - T^i_q x_p. \tag{5}$$

Let us note that formula (5) describes the total angular momentum of the system, and not the orbital one (7). The decomposition of the total angular momentum into spin and orbital parts occurs when the canonical energy-momentum tensor is used, which, generally speaking, is not symmetric. It is possible that defining the total angular momentum of a system by means of formula (5) will prove convenient in general relativity, since the symmetric energy-momentum tensor can be taken from Einstein' s equations (in particular, apparently, one may use the Einstein tensor G^{ik} instead of T^{ik}).

Let us now consider the group of motions of a 4-dimensional space of constant curvature (the de Sitter group). In stereographic coordinates the line element of this space has the form (8) $-ds^2 = \varphi^2 dx^{i2}$, where $\varphi = (1 + (r^2 - x_0^2)/4R^2)^{-1}$ $(r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2)$. The de Sitter group is isomorphic to the rotation group O_5. Its generators may be represented in the form (8)

$$\Pi_l = \varphi^{-1} P_l + \frac{x_k}{2R^2} L_{kl}, \quad L_{ik} = x_i P_k - x_k P_i, \quad \text{where } P_k = -i\,\partial/\partial x^k.$$

Thus, only in the limit of flat space ($R \to \infty$) does the 4-momentum operator Π_l acquire the meaning of a coordinate-translation operator. In curved space the "momentum" will always be mixed with the "angular momentum." To avoid this, it would be necessary to choose in the group of motions 4 mutually commuting operators and call them the 4-momentum operators. However, this is impossible, since the de Sitter group is simple and has rank 2. As a result, the 4-momentum and the total angular momentum of the system become components of the 5-dimensional total angular momentum of the system.

Let us write down the integral conserved quantities, assuming, as usual, that at spatial infinity $T^{ik} = 0$.* For the components of the 4-dimensional angular momentum we obtain the usual expression

$$M_{lm} = \int M^0_{(lm)} \varepsilon(N) N_0 d_3 v = \int \varphi^4 M^0_{(lm)}\, d^3x, \tag{6}$$

where $M^0_{(lm)}$ is determined by formula (5), but x are no longer Cartesian, but stereographic coordinates.

The generalized "translations" give 4 Killing vectors, which define (under the same conditions at spatial infinity) the generalized "4-momentum" with components:

$$P^0 = \int \sqrt{-g}\, T^0_k \xi^k_0\, d^3x =$$

$$= \int \varphi^3 T^0_0 d^3x - \int \frac{r^2}{2R^2} \varphi^4 T^0_0 d^3x + \int \varphi^4 \frac{x_0 x_\alpha}{2R^2} T^0_\alpha d^3x \quad (\alpha = 1, 2, 3); \tag{7}$$

$$P^\alpha = \int \sqrt{-g}\, T^0_k \xi^k_\alpha d^3x =$$

$$= \int \varphi^3 T^0_\alpha d^3x - \int \varphi^4 \frac{\sum_{k\neq\alpha} x_k^2}{2R^2} T^0_\alpha d^3x + \int \varphi^4 \sum_{k\neq\alpha} T^0_k \frac{x_k x_\alpha}{2R^2} d^3x. \tag{8}$$

These expressions have meaning in the case where the metric of space-time is regarded as rigidly prescribed and is not connected with the distribution of matter. If, however, a geometrized theory is considered (general relativity), then T^{ik} will necessarily be proportional

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* We consider a system of bodies occupying a finite volume in de Sitter space. The metric V_4 is regarded as prescribed.

metric tensor g^{ik}, and instead of (6)–(8) we obtain

$$M_{0\alpha} = \int \sqrt{-g}\,\xi^0_{(0\alpha)} d^3x = \int \varphi^4 x_\alpha d^3x \tag{9}$$

(the other components of the "moment" are equal to zero),

$$P^0 = \int \varphi^3 d^3x - \int \frac{r^2}{2R^2}\varphi^4 d^3x = \int \frac{1-(r^2+x_0^2)/4R^2}{1+(r^2-x_0^2)/4R^2}\,\varphi^3 d^3x; \tag{10}$$

$$P^\alpha = \int \frac{x_0 x_\alpha}{2R^2\left(1+(r^2-x_0^2)/4R^2\right)}\varphi^3 d^3x. \tag{11}$$

Let us note that if ${T_k}^2$ can be reduced to a divergence, the integrals (7), (8) become surface integrals, which in a closed model leads to $P^i = 0$.

The expressions obtained for the integral conserved quantities possess two important features. The first of these is chronometric invariance (ch.i.). In the case under consideration, the lines of time are orthogonal to the space-time section $x^0 = \text{const}$. Therefore the ch.i. element of three-dimensional volume (coinciding with the invariant volume element d_3V according to Synge ([4])) has the form $\varphi^3 d^3x$; the normal vector is $N_i = \delta_i^0\sqrt{g_{00}}$; $N_0 = \sqrt{g_{00}}$; $N^0 = 1/\sqrt{g_{00}}$. Taking this into account, let us write the expressions for P^i, M_{ik} in explicitly chronometrically invariant form (the ch.i. components of the tensors are integrated over the ch.i. volume element)

$$P^0 = \int \varphi^3 d^3x - \int \frac{x^\alpha M_{0(0\alpha)}}{2\sqrt{g_{00}}R^2}\,\varphi^3 d^3x,$$

$$P^\alpha = \int \frac{T_{0\alpha}}{\sqrt{g_{00}}}\varphi^3 d^3x - \int \frac{x^k M_{0(\alpha k)}}{2\sqrt{g_{00}}R^2}\,\varphi^3 d^3x, \qquad M_{lm} = \int \frac{M_{0(lm)}}{\sqrt{g_{00}}}\varphi^3 d^3x.$$

Ch.i. in Zelmanov' s sense ensures the independence of the quantities found from arbitrary coordinate transformations connected with the given reference system. Thus these quantities do indeed characterize the chosen system. The motion of the system as a whole is described here by the motion group V_4, with respect to which the set of 10 components P^i, M_{ik} forms a regular representation. "Translations" in V_4 mix P^i and M_{ik} with one another.

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Note: Figure translations are in progress. See original paper for figures.