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Abstract

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MATHEMATICS

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ON ONE NECESSARY CONDITION FOR TIKHONOV REGULARIZABILITY

(Presented by Academician A. N. Tikhonov on 17 IV 1970)

Suppose we have a mapping A of a metric space into a metric space U, with $Az_1 \neq Az_2$ if $z_1 \neq z_2$. The problem of approximate solution of the equation

$$Az = u \tag{1}$$

is called Tikhonov-regularizable ([1]) if there exists a one-parameter family of mappings $R_\delta : U \to Z,\ 0 < \delta \leq 1$, for which

$$\lim_{\delta \to 0} \sup_{\{u \in U:\ \rho(u, Az) \leq \delta\}} \rho(z, R_\delta u) = 0, \quad \forall z \in Z. \tag{2}$$

Theorem. If problem (1) is Tikhonov-regularizable and N is an everywhere dense set in U, then

$$\bigcup_{n=1}^{\infty} \{R_{1/n} N\}$$

is an everywhere dense set in Z.

Proof. For an arbitrary point $z \in Z$ and any natural n, take $u_n \in N$ such that $\rho(Az, u_n) \leq 1/n$. By the regularizability condition (2),

$$\lim_{n \to \infty} \rho(z, R_{1/n} u_n) = 0,$$

i.e. the sequence $\{R_{1/n} u_n\}$ converges to the element z of the space Z. Since $u_n \in N$, the theorem is proved.

Corollary. If problem (1) is Tikhonov-regularizable and in the metric space U there exists an infinite everywhere dense set of cardinality τ, then in the metric space Z there also exists an everywhere dense set of cardinality τ. In particular, separability of Z follows from separability of U.

From this, examples of nonregularizable problems easily follow.

1. For the embedding operator of the space of measurable bounded functions on the segment $M_{[0,1]}$ into $L_{2[0,1]}$, problem (1) is not Tikhonov-regularizable.
2. Let Z be a nonseparable metric space of monotonically increasing functions on $[0,1]$, the distance between which is measured in the metric of the space $M_{[0,1]}$; let U be an arbitrary separable metric space (for example L_2). Then for any mapping $A : Z \to U$, regularization is impossible, i.e. it is impossible to construct approximate solutions $z_\delta \in Z$ whose deviation from the exact solution $z \in Z$ would tend to zero in the metric of the space $M_{[0,1]}$. Note that A may be a linear continuous or a completely discontinuous operator.
3. The integral equation

$$\int_0^1 k(x,t)\, dg(t) = u(x),$$

considered from $M_{[0,2\pi]}$ into $L_{2[0,2\pi]}$, is nonregularizable for any kernel $k(x,t)$.

4. The integral equation

$$Az = \int_0^{2\pi} k(x-t) z(t)\, dt = u(x),$$

considered from $M_{[0,\,2\pi]}$ into $L_{2[0,\,2\pi]}$, where $k(t)$ is a 2π-periodic even function satisfying the condition

$$\int_0^{2\pi} k(t) \cos nt\, dt \neq 0 \qquad (n = 0, 1, 2, \ldots),$$

is not regularizable. With a suitable choice of $k(t)$ (for example, $k(t) \in L_{2[0,\,2\pi]}$) the linear operator A will be completely continuous.

The last result shows that A. B. Bakushinskii' s question (see [2], p. 71) on the possibility of Tikhonov regularization of a linear bounded or completely continuous operator from a Banach space into a Banach space, under the condition of uniqueness of the solution of equation (1), must be answered in the negative. However, the question of regularizability remains open for a separable Banach space Z.

From the corollary to the theorem, together with A. B. Bakushinskii' s result on the regularizability of problem (1), if Z is a uniformly convex Banach space, A is a linear bounded operator, $A^{-1}0 = \{0\}$, and U is a Banach space, we obtain an interesting corollary for linear operators in a Banach space: if there exists a linear continuous mapping A of a uniformly convex Banach space Z into a separable Banach space U, $A^{-1}0 = \{0\}$, then Z is separable.

If there exists a completely continuous linear mapping A of a uniformly convex Banach space Z into a normed space, such that $A^{-1}0 = \{0\}$, then Z is separable. This follows from the separability of the range of a completely continuous operator.

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CITED LITERATURE

[1] A. N. Tikhonov, DAN, **153**, No. 1, 49 (1963). [2] A. B. Bakushinskii, in the collection *Computational Methods and Programming*, issue 12, Moscow, 1969, p. 56.

Note: Figure translations are in progress. See original paper for figures.