UNIFORM ESTIMATE FOR THE NUMBER OF REPRESENTATIONS OF UNITY BY A BINARY FORM OF DEGREE \(n \ge 3\)

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Abstract

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UNIFORM ESTIMATE FOR THE NUMBER OF REPRESENTATIONS OF UNITY BY A BINARY FORM OF DEGREE $n \geq 3$

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Let Z be the set of all integers; let $F_n(x, y)$ be a binary form of the n-th degree in x and y over Z; let $\Phi(F_n)$ be the number of solutions in integers x and y of the indeterminate equation

$$F_n(x, y) \equiv a_0 x^n + \dots + a_{kx}^{n-k} y^k + \dots + a_{ny}^n = 1, \quad \text{where } a_0, \dots, a_n \in Z. \tag{1}$$

For the case of irreducibility over Z of the form F_n, the following is known about the quantity $\Phi(F_n)$. A. Thue ([1]) proved that for $n \geq 3$, $\Phi(F_n)$ is finite; using his method, one can find an effective upper estimate for the quantity $\Phi(F_n)$ for each particular form F_n (see, for example, ([2]), § 60, written by the author of this article). However, this estimate is a function of the coefficients of the form F_n and does not give a single general numerical upper bound for all $\Phi(F_n)$ in the aggregate for a given n. The effective estimate established by A. Baker for the coordinates x and y of the solutions themselves did not change the situation in the question of the number of solutions.

Meanwhile, B. N. Delone ([2,4,5]) proved that for all cubic forms F_3 irreducible over Z with discriminant $\Delta(F_3) < 0$, the number of solutions of equation (1) has a finite upper bound common to all these forms, equal to 5. For cubic forms with $\Delta(F_3) > 0$, C. L. Siegel ([6]) established an analogous estimate, equal to 18, for all such forms whose discriminant exceeds a certain constant C. D. K. Faddeev (see in ([2]) § 70, written by him) lowered this estimate from 18 to 15, and for the constant C reported an approximate value equal to 10^{33}. Since the number of forms F_3 with positive discriminant less than C is finite, for them, by the method described above, one can establish a common upper bound for the number of solutions of equation (1) when $n = 3$. Therefore there exists a finite common bound for $\Phi(F_3)$ for all F_3 irreducible over Z, although this bound has still not been computed by anyone. The aim of the present work is to establish that, with the exception of trivial and obvious exceptions, such a common upper bound for $\Phi(F_n)$ exists for all F_n with fixed n greater than 3.

Theorem 1. *The number $\Phi(F_n)$ of integral representations of unity by each binary form $F_n(x, y)$ over Z, irreducible over Z, of fixed degree n greater than 3, satisfies the inequality*

$$\Phi(F_n) \leq \varphi(n) = 235\,n^6. \tag{2}$$

From this main theorem it is easy to derive:

Theorem 2. *If a binary form $F_n(x, y)$ over Z of fixed degree n, with $n \geq 3$, is not a power of a linear form or of an indefinite quadratic form $f(x, y)$ over Z integrally representing unity, then the number $\Phi(F_n)$ of integral representations of unity by this form F_n is bounded above as in Theorem 1 (see (2)).*

The estimate $235\,n^6$ for $\varphi(n)$ can be substantially reduced in various ways. However, it is evidently impossible to replace $\varphi(n)$ by an absolute constant independent of n.

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Note: Figure translations are in progress. See original paper for figures.