



---

Soviet-era science, translated into English

# ON POLYHEDRAL REPRESENTATIONS OF METRIC SPACES

MATHEMATICS

1970

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-197001.77245>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 513.83

**MATHEMATICS**

**I. M. KOZLOVSKII**

## **ON POLYHEDRAL REPRESENTATIONS OF METRIC SPACES**

*(Presented by Academician P. S. Aleksandrov on 28 VIII 1969)*

By a polyhedron in this note is meant a simplicial complex in which the distance is defined as the maximum of the differences of the barycentric coordinates. Following the terminology adopted in <sup>(1,2)</sup>, we say that a mapping  $g : X \rightarrow K$  into a complex is an admissible modification of a mapping  $f : X \rightarrow K$  if  $fx$  is contained in the closed star of the point  $gx$  for every  $x \in X$ ; a mapping  $f$  is irreducible if  $fX \subseteq gX$  for every admissible modification  $g$  of it; a mapping  $\pi : K \rightarrow K'$  is normal if  $\pi$  maps the complex  $K$  simplicially into some multiple barycentric subdivision of the complex  $K'$ . An inverse spectrum  $\{P_i, \pi_{ij}\}$  is called a polyhedral representation of the space  $X$  if  $X = \lim\{P_i, \pi_{ij}\}$ , the  $P_i$  are polyhedra, and the  $\pi_{i+1;i}$  map  $P_{i+1}$  simplicially onto some subdivision of the complex  $P_i$ ; the representation is called irreducible if the projections  $\pi_i : X \rightarrow P_i$  are irreducible, and standard if the mappings  $\pi_{i+1;i}$  are normal and the polyhedra  $P_i$  are finite-dimensional.

As Isbell showed <sup>(3)</sup>, every complete metric space has an irreducible standard polyhedral representation. For compacta this assertion was proved by Freudenthal <sup>(1)</sup> (see also <sup>(6)</sup>). B. Pasynkov <sup>(5)</sup> proved that every  $n$ -dimensional compactum can be represented as the limit of an inverse spectrum of  $n$ -dimensional polyhedra with finite-to-one projections "onto." In the present note generalizations of these results (Theorems 1, 2, 3) are presented, as well as some facts concerning uniformly zero-dimensional mappings into polyhedra.

Let  $\mathcal{F}$  be a system of closed subsets of the space  $X$ . A polyhedral representation  $\{P_i, \pi_{ij}\}$  of the space  $X$  will be called a (irreducible, standard) polyhedral representation of the system  $\mathcal{F}$  if for every  $F \in \mathcal{F}$  there is a natural number  $j$  such that  $\{\pi_i(F), \pi_{ij}; i = j, j+1, \dots\}$  is a (irreducible, standard) polyhedral representation of the subspace  $F \subseteq X$ . The symbol  $\omega_f K$  will denote the covering of the space  $X$  by the inverse images of the principal stars of the complex  $K$  under the mapping  $f : X \rightarrow K$ . If the covering  $\omega_f K$  is locally finite, then the mapping  $f$  will be called locally finite, in order to emphasize the fact that each point  $x \in X$  has a neighborhood that is mapped by  $f$  into a finite subcomplex of the complex  $K$ .

**Lemma 1.** *Let  $f$  be a locally finite mapping of a normal space  $X$  into a complex  $K$ , and let  $C$  be a closed subset of  $X$ . Every (irreducible) admissible modification  $g_1$  of the mapping  $f_1 = f|_C$  can be extended to  $X$  to an (irreducible) admissible modification  $g$  of the mapping  $f$ .*

**Lemma 2.** *Let  $\mathcal{F}$  be a locally finite system of closed subsets of a paracompact space  $X$ . Every mapping of the space*

---

\* By a mapping, everywhere, a continuous mapping is meant.

of  $X$  into a complex has an admissible modification that is irreducible on each element of the system  $\mathcal{F}$ .\*

**Theorem 1.** Every  $\sigma$ -locally finite system of closed subsets of a complete metric space has an irreducible standard polyhedral representation.

The proof is similar to the proof of Isbell's theorem (2, 3) and uses the assertion of Lemma 2.

By analogous methods one proves

**Theorem 2.** Let  $X$  be a complete metric space; let  $a_i$  be a countable sequence of its open covers and let  $\mathcal{F}$  be a  $\sigma$ -locally finite system of closed subsets in  $X$ . There exists an irreducible polyhedral representation  $\{P_i, \pi_i^{i+1}\}$  of the system  $\mathcal{F}$  such that the projections  $\pi_i : X \rightarrow P_i$  are canonical  $a_i$ -maps.\*\*

It is easy to give an example showing that, for  $\sigma$ -locally countable systems, Theorems 2 and 3 are, generally speaking, not true. However, for every  $\sigma$ -locally countable system  $\mathcal{F}$  of closed sets of a paracompact space there exists a  $\sigma$ -discrete system  $\Phi$  of closed sets such that each element of  $\mathcal{F}$  is a countable union of some elements of  $\Phi$ .

**Lemma 3.** Let  $f : X \rightarrow K$  be a uniformly zero-dimensional mapping of a metric space  $X$  into an  $n$ -dimensional complex  $K$ . For every  $\varepsilon > 0$  there exist an  $n$ -dimensional complex  $K_1$  and mappings  $\pi : K_1 \rightarrow K$ ,  $f_1 : X \rightarrow K_1$ , satisfying the following conditions: 1) the mapping  $\pi$  is normal and does not degenerate simplices; 2) the mapping  $f_1$  is uniformly zero-dimensional,  $f = \pi f_1$ , and the cover  $\omega_{f_1} K_1$  is an  $\varepsilon$ -cover; 3) if the cover  $\omega_{fK}$  is locally finite, then the cover  $\omega_{f_1} K_1$  may also be regarded as locally finite.

**Proof.** From the uniform zero-dimensionality of the mapping  $f$  it follows that, for some natural number  $s$ , the inverse images of the principal stars of the  $s$ -fold barycentric subdivision  $K^{(s)}$  of the complex  $K$  (discretely)  $\varepsilon$ -split. Let  $\{U_{\alpha\gamma}\}$  be open sets into which the set  $U_\alpha \in \omega_{fK}^{(s)}$  (discretely)  $\varepsilon$ -splits. As the complex  $K_1$  we take the nerve of the cover  $\{U_{\alpha\gamma}\}_{\alpha,\gamma}$ . The correspondence  $U_{\alpha\gamma} \rightarrow U_\alpha$  uniquely determines a simplicial mapping  $\pi : K_1 \rightarrow K^{(s)}$ . Since the sets  $U_{\alpha\gamma}$ , for fixed  $\alpha$ , are pairwise disjoint, different vertices of a simplex  $t \subset K_1$  correspond to

different vertices of the simplex  $\pi(t)$ , and, consequently,  $\pi$  does not degenerate simplices.

Let  $\{f_\alpha\}$  be a partition of unity on  $X$  corresponding to the mapping  $f : X \rightarrow K^{(s)}$ . Putting  $f_{\alpha\gamma} = f_\alpha$  on  $U_{\alpha\gamma}$ ,  $f_{\alpha\gamma} = 0$  on  $X \setminus U_{\alpha\gamma}$ , we define a new partition of unity  $\{f_{\alpha\gamma}\}_{\alpha,\gamma}$ , which defines a uniformly zero-dimensional mapping  $f_1 : X \rightarrow K_1$  and the cover  $\omega_{f_1} K_1$ , coinciding with the cover  $\{U_{\alpha\gamma}\}_{\alpha,\gamma}$ . The latter, by construction, is an  $\varepsilon$ -cover. If now the cover  $\omega_{fK}$ , and hence also the cover  $\omega_{fK}^{(s)}$ , is locally finite, and the system  $\{U_{\alpha j}\}_\gamma$  is discrete for every  $\alpha$ , which can always be achieved by a suitable choice of the number  $s$ , then the cover  $\omega_{f_1} K_1$  is also locally finite. The lemma is proved.

**Proposition 1.** Let  $f : X \rightarrow K$  be a uniformly zero-dimensional mapping of a metric space  $X$  into a complex  $K$ , and let  $\omega$  be an open cover of the space  $X$ . There exist a complex  $K_\omega$  and mappings  $\pi : K_\omega \rightarrow K$ ,  $f_\omega : X \rightarrow K_\omega$ , satisfying the following conditions: 1)  $f_\omega$  is a uniformly zero-dimensional canonical  $\omega$ -mapping; 2)  $\pi f_\omega = f$ ; 3) the mapping  $\pi$  is piecewise affine and nondegenerate; 4) if  $f$  is locally finite, then  $f_\omega$  may also be regarded as locally finite.

Proposition 1 follows from Lemma 3. As a consequence of Proposition 1 we obtain

\* The mapping  $f$  is irreducible on  $A \subset X$  if the mapping  $f|_A$  is irreducible.

\*\* A mapping  $f : X \rightarrow K$  is called a canonical  $a$ -mapping if the cover  $\omega_{fK}$  is inscribed in the cover  $a$ .

**Proposition 2.** Every uniformly zero-dimensional mapping of an  $n$ -dimensional\* metric space into an  $n$ -dimensional complex is locally essential.\*\*

**Lemma 4.** Let  $f$  be a uniformly zero-dimensional mapping of a metric space into a complex  $T$  with boundary  $S$ . If  $\dim X < \dim T$ , then there exists a uniformly zero-dimensional mapping  $g : X \rightarrow S$  that coincides with  $f$  on  $f^{-1}S$ .

**Proposition 3.** Every uniformly zero-dimensional locally finite mapping of a metric space into a finite-dimensional complex has a uniformly zero-dimensional irreducible admissible alteration.

Proposition 3 follows from Proposition 2 and Lemma 4. The requirement of local finiteness is essential here.

**Proposition 4.** Let  $f$  be a mapping of a closed subset  $A$  of a metric space  $X$  into an  $n$ -dimensional cube  $I$ . If  $\dim X \setminus A \leq n$ , then the set of mappings, each of which is uniformly zero-dimensional outside any  $\varepsilon$ -neighborhood of  $A$ , is a dense set of type  $G_\delta$  in the space  $C_f(X, I)$  of mappings  $g : X \rightarrow I$  that coincide with  $f$  on  $A$ ; if, in addition,  $f$  is uniformly zero-dimensional, then the uniformly zero-dimensional mappings form a dense set of type  $G_\delta$  in the space  $C_f(X, I)$ .

**Remark.** As the corresponding example shows, in the case of an arbitrary  $f$  the set of uniformly zero-dimensional mappings  $g \in C_f(X, I)$  may be empty.

With the aid of Lemma 3, Proposition 3, and Katetov's theorem <sup>(4)</sup>, one can prove the following principal result.

**Theorem 3.** Every  $\sigma$ -locally finite system of closed subsets of a complete finite-dimensional metric space  $X$  has an irreducible standard polyhedral representation  $\{P_i, f_{ij}\}$  with nondegenerate mappings  $f_{ij}$  and uniformly zero-dimensional projections  $f_i : X \rightarrow P_i$ . If  $\{\omega_i\}$  is a countable sequence of open covers of  $X$ , then the projections  $f_i$  may be taken to be canonical  $\omega_i$ -mappings.

**Remark.** If  $X$  is compact, then the projections  $f_{ji}$  may be taken to be finite-to-one.

Dimension characteristics are known by means of: 1) essential mappings of normal spaces into a cube (P. S. Aleksandrov); 2) irreducible  $\omega$ -mappings of normal spaces into polyhedra (P. S. Aleksandrov, Dowker); 3) uniformly zero-dimensional mappings of metric spaces into a cube (Katetov, for compacta<sup>\*\*\*</sup> –Hurewicz). With the aid of Propositions 1, 2, 3 these characteristics can be generalized as follows.

**Theorem 4.** For metric spaces the following conditions are equivalent:

- 1)  $\dim X = n$ ;
- 2) the space  $X$  possesses a uniformly zero-dimensional essential mapping into an  $n$ -dimensional cube;
- 3) every uniformly zero-dimensional mapping of the space  $X$  into an  $n$ -dimensional cube is locally essential;
- 4) for every open cover  $\omega$  of it, the space  $X$  possesses an irreducible uniformly zero-dimensional canonical  $\omega$ -mapping onto an  $n$ -dimensional complex.

The equivalence of conditions (1) and (2) for arbitrary metric spaces, and also of conditions (1), (2), and (3) for compacta, was proved by A. Zarelua and Yu. Smirnov <sup>(8)</sup>. The existence of uniformly zero-dimensional  $\omega$ -mappings (without the assumption of irreducibility) of an  $n$ -dimensional metri-

---

\* By dimension here and below is meant the dimension  $\dim$ .

\*\* A mapping  $f$  into an  $n$ -dimensional complex  $T$  is called locally essential if some smaller  $n$ -dimensional simplex contained in  $T$  is covered essentially <sup>(8)</sup>.

\*\*\* For compacta, uniformly zero-dimensional mappings coincide with zero-dimensional ones.

of a metric space into an  $n$ -dimensional polyhedron (and even a more general assertion) was proved earlier by E. G. Sklyarenko (<sup>(7)</sup>, Theorem K). Concerning Proposition 4 see also <sup>(9)</sup>.

I take this opportunity to express my sincere gratitude to my supervisor B. A. Pasynkov for his constant attention and great help in the work.

Moscow Institute of Physics and Technology

Received  
25 VI 1969

## REFERENCES

- <sup>1</sup> H. Freudenthal, *Composito Math.*, 4, No. 2 (1937).
- <sup>2</sup> J. R. Isbell, *Proc. Koninkl. Nederl. Akad. Wet., Ser. A*, 64, No. 2 (1961).
- <sup>3</sup> J. R. Isbell, *Pacific J. Math.*, 12, No. 1 (1962).
- <sup>4</sup> M. Katetov, *DAN*, 79, No. 2, 189 (1951).
- <sup>5</sup> B. A. Pasyukov, *Tr. Mosk. matem. obshch.*, 13, 136 (1965).
- <sup>6</sup> E. G. Sklyarenko, *DAN*, 134, No. 4 (1960).
- <sup>7</sup> E. G. Sklyarenko, *UMN*, 21, issue 4 (1966).
- <sup>8</sup> A. V. Zarelua, Yu. S. Smirnov, *DAN*, 148, No. 5 (1963).
- <sup>9</sup> S. Sakai, *Proc. Japan Acad.*, 44, 939 (1968).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*