# A METHOD FOR SOLVING SOME BOUNDARY-VALUE PROBLEMS IN THE STABILITY THEORY OF CYLINDRICAL SHELLS

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## Abstract

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# A METHOD FOR SOLVING SOME BOUNDARY-VALUE PROBLEMS IN THE STABILITY THEORY OF CYLINDRICAL SHELLS

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Most problems on the stability of an axisymmetric state of equilibrium of a circular cylindrical shell reduce (after the usual separation of variables) to a homogeneous boundary-value problem for a system of ordinary differential equations of the form

$$\begin{gathered} P_1(D)V + [(2 - c_{11}D\chi)D^2 + v_{12}D^3\chi]\,W = 0, \\ [(2 - c_{11}D\chi)D^2 + c_{12}D^3\chi]\,V - [P_2(D) + c_{22}D\chi]\,W = 0. \end{gathered} \tag{1}$$

Here $V(\xi)$ and $W(\xi)$ are the sought functions of the problem; $\xi$ is the dimensionless axial coordinate; $P_1$ and $P_2$ are differential operators of the 4th order with constant coefficients, considered as polynomials of degree 4 in the differential operator $D = d/d\xi$; $\chi(\xi)$ is the solution of the boundary-value problem for the ordinary differential equation of the 4th order

$$D^4\chi + 4c_{01}D^2\chi + 4\chi = 0 \quad (c_{01}^2 < 1), \tag{2}$$

which describes the subcritical axisymmetric bending of a cylindrical shell; $c_{ij}$ are constant coefficients. In this case the function $\chi$, the coefficients $c_{ij}$, and also the coefficients of the differential operators $P_1$ and $P_2$ may depend on several parameters $q_1, q_2, \dots, q_n$, which determine the field of external forces.

Let us first consider a semi-infinite shell. In this case the solution of the boundary-value problem for equation (2) has the form

$$\chi(\xi) = \sum_{p=1}^{2} c_p z_p(\xi) \quad (\xi \geq 0), \tag{3}$$

where $z_p(\xi) = \exp(\lambda_p \xi)$; $\lambda_p$ are complex-conjugate roots of the equation

$$\lambda^4 + 4c_{01}\lambda^2 + 4 = 0 \tag{4}$$

with negative real parts; $c_p$ is a pair of complex-conjugate constants, which are determined from the inhomogeneous boundary conditions characteristic of the subcritical equilibrium of the shell.

Thus, system (1) is a system of equations with exponential coefficients. We seek its particular solution in the form

$$V_j(\xi) = \left(1 + \sum_{p=1}^{2} a_{jp} z_p + \sum_{p=1}^{2}\sum_{q=1}^{2} a_{jpq} z_{pz} q + \ldots\right) \exp(\Lambda_j \xi),$$

$$W_j(\xi) = \left(b_j + \sum_{p=1}^{2} b_{jp} z_p + \sum_{p=1}^{2}\sum_{q=1}^{2} b_{jpq} z_{pz} q + \ldots\right) \exp(\Lambda_j \xi), \tag{5}$$

where $\Lambda_j, b_j, a_{jp}, b_{jp}, a_{jpq}, b_{jpq}, \ldots$ are, generally speaking, unknown complex constants.

Substituting (3) and (5) into (1) and equating to zero the coefficients of exponentials with identical exponents, we arrive at an infinite algebraic system of the form

$$P_1(\Lambda_j) + 2\Lambda_j^2 b_j = 0, \qquad 2\Lambda_j^2 - P_2(\Lambda_j) b_j = 0; \tag{6a}$$

$$P_1(\Lambda_{jp}) a_{jp} + 2\Lambda_{jp}^2 b_{jp} = (c_{11}\Lambda_j^2 - c_{12}\lambda_p^2) b_j \lambda_p c_p, \tag{6b}$$

$$2\Lambda_{jp}^2 a_{jp} - P_2(\Lambda_{jp}) b_{jp} = (c_{11}\Lambda_j^2 - c_{12}\lambda_p^2 + c_{22} b_j) \lambda_p c_p;$$

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$$P_1(\Lambda_{jpq}) a_{jpq} + 2\Lambda_{jpq}^2 b_{jpq} = (c_{11}\Lambda_{jp}^2 - c_{12}\lambda_q^2) b_{jp} \lambda_q c_q, \tag{6c}$$

$$2\Lambda_{jpq}^2 a_{jpq} - P_2(\Lambda_{jpq}) b_{jpq} = [(c_{11}\Lambda_{jp}^2 - c_{12}\lambda_q^2) a_{jp} + c_{22} b_{jp}] \lambda_q c_q;$$

Here $p, q, \ldots$ take values from 1 to 2, $\Lambda_{jp\ldots} = \Lambda_j + \lambda_p + \cdots$.

As $\Lambda_j$ we take a root of the eighth-degree equation

$$P(\Lambda) \equiv P_1(\Lambda) P_2(\Lambda) + 4\Lambda^4 = 0, \tag{7}$$

and determine $b_j$ as

$$b_j = -P_1(\Lambda_j)/2\Lambda_j^2. \tag{8}$$

Thus the system (6a) will be identically satisfied. For known $\Lambda_j, b_j$ and under the condition $P(\Lambda_{jp}) \neq 0$, from (6b) we find the coefficients $a_{jp}$ and $b_{jp}$, then—under the condition $P(\Lambda_{jpq}) \neq 0$—we find $a_{jpq}$ and $b_{jpq}$, and so on. In this way all the coefficients of the series (5) are successively determined.

If equation (7) has no multiple roots, then the general solution of the system (1) can be represented in the form

$$V = \sum_{j=1}^{8} A_j V_j, \qquad W = \sum_{j=1}^{8} A_j W_j, \tag{9}$$

where $A_j$ are arbitrary constants. Subjecting (9) to homogeneous boundary conditions, we obtain a homogeneous system of 8 linear equations in the 8 unknown coefficients $A_j$. From the characteristic equation of this system, of the form

$$\Omega(q_1, q_2, \dots, q_n) = 0 \tag{10}$$

we determine the eigenvalues of the parameters $q_1, q_2, \dots, q_n$, in other words, the critical values of the external forces.

When the conditions

$$P(\Lambda_{jp}) \neq 0, \qquad P(\Lambda_{jpq}) \neq 0, \dots$$

are satisfied, the series (5) converge uniformly on the half-axis $\xi \geq 0$. Indeed, since the numbers $\Lambda_j$ are bounded in modulus, by means of the relations (6) one can establish that the remainders of the series (5) are majorized by the convergent series

$$\sum_{n=N}^{\infty} \frac{2^n r_n}{n^n} \qquad (0 \leq r_n < \infty)$$

with $N$ independent of $\xi$.

In the case of a shell of finite length it is necessary to use all the roots of equation (4). Let $\lambda_1$ and $\lambda_2$ be complex conjugate roots of this equation with negative real parts, and $\lambda_3$ and $\lambda_4$ with positive real parts. Then the general solution of equation (2) can be represented in the form

$$\chi(\xi) = \sum_{p=1}^{4} c_p z_p(\xi).$$

Here $0 \le \xi \le \beta = [3(1-\nu^2)]^{1/4}(Rh)^{-1/2}L$ ($L$ is the shell length, $R$ its radius, $h$ its thickness, $\nu$ Poisson' s ratio), $c_p$ $(p = 1, 2, 3, 4)$ are two pairs of complex conjugate constants,

$$z_p(\xi) = \exp(\lambda_p \xi) \quad (p = 1, 2), \qquad z_p(\xi) = \exp[\lambda_p(\xi - \beta)] \quad (p = 3, 4).$$

In what follows we require that the condition

$$\exp(-\beta \operatorname{Re} \lambda_p) \ll 1 \qquad (p = 3, 4) \tag{11}$$

be satisfied (it may fail only for very short shells). Then we shall have

$$\max|z_p z_{q+2}| \ll \max|z_p z_q| = \max|z_{p+2} z_{q+2}| = 1,$$

$$\|z_p z_{q+2}\| \ll \|z_p z_q\| = \|z_{p+2} z_{q+2}\| \qquad (p, q = 1, 2). \tag{12}$$

Here $\max|\ldots|$ and $\|\ldots\|$ denote the maximum value of the modulus and the norm on the interval $0 \le \xi \le \beta$. Under conditions (12), system (1) can be satisfied by the substitution

$$V_j = \left[1 + \left(\sum_{p=1}^{2} + \sum_{p=3}^{4}\right) a_{jp} z_p + \left(\sum_{p=1}^{2}\sum_{q=1}^{2} + \sum_{p=3}^{4}\sum_{q=3}^{4}\right) a_{jpq} z_{pz} q + \ldots\right] \exp(\Lambda_j \xi),$$

$$(13)$$

$$W_j = \left[b_j + \left(\sum_{p=1}^{2} + \sum_{p=3}^{4}\right) b_{jp} z_p + \left(\sum_{p=1}^{2}\sum_{q=1}^{2} + \sum_{p=3}^{4}\sum_{q=3}^{4}\right) b_{jpq} z_{pz} q + \ldots\right] \exp(\Lambda_j \xi),$$

where $\Lambda_j, b_j, a_{jp}, b_{jp}, a_{jpq}, b_{jpq}, \ldots$ are determined from relations (6)–(8), in which the indices $p, q, \ldots$ now take values from 1 to 4. As before, the general solution of system (1) has the form (9), and the critical values of the external forces are determined from condition (10).

Such an analogy with the case of a semi-infinite shell is obtained only thanks to condition (11). Dropping it leads to the fact that each of the equations of system (6) will contain an infinite number of unknowns, and the process described above of successively determining the coefficients of the expansions (5) cannot be carried out.

As before, uniform convergence of the series (13) on the interval $0 \leq \xi \leq \beta$ can be established. Only now the majorant series will have the form

$$\sum_{n=N}^{\infty} \frac{2^{n+1} r_n}{n^n}.$$

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*Note: Figure translations are in progress. See original paper for figures.*