# ON SUMS OF MULTIPLICATIVE FUNCTIONS

B. V. LEVIN, N. M. TIMOFEEV

1970

SovietRxiv

## Abstract

## Full Text

UDC 511.3

## MATHEMATICS

**B. V. LEVIN, N. M. TIMOFEEV**

# ON SUMS OF MULTIPLICATIVE FUNCTIONS

*(Presented by Academician Yu. V. Linnik on 12 I 1970)*

Recently G. Halász ([1]) successfully applied an analytic method to the estimation of mean values of multiplicative functions. Combining his ideas with new considerations, we obtain some results concerning sums of multiplicative functions. To characterize these results, we give a special case of Theorem 2 proved below:

**Theorem 1.** *If $f(p^r) = O(1)$ and there exist $\tau > 1/2$, $A = \mathrm{const}$ such that*

$$s(x) = \sum_{p \le x} \frac{|f(p)| \log p}{p} \le \tau \log x + A \frac{\log x}{(\log\log x)^{1+\varepsilon}}, \tag{1}$$

*then there exist constants $C_0$, $a$ and a slowly varying function $L(u)$ ($|L(u)| = 1$) such that*

$$\sum_{n \le x} f(n) = C_0 x^{1+ia} L(\log x)^{\tau-1} x + o(x \log^{\tau-1} x). \tag{2}$$

Below we give an outline of the proof of a more general result, which contains Theorem 1 as a special case.

**Theorem 2.** *Let $f(n)$ be a multiplicative function satisfying condition (1) with $\tau > 1/2$,*

$$|f(p^r)| \le \frac{1}{3} p^{r\gamma} \tag{3}$$

*for $r \ge 1$ and some $\gamma = 1/2 - \varepsilon$, $\varepsilon > 0$. Then, if*

$$\sum_{y < p \le x} \frac{|f(p)| \log p}{p} = O\left(\log \frac{x}{y}\right) + \varepsilon(y), \tag{4}$$

*where $\varepsilon(y)$ tends monotonically to zero as $y \to \infty$, or*

$$f(p) \geq c > 0, \qquad \sum_{p \leq x} f(p) \log p = O(x), \tag{5}$$

*then there exist $C$, $a$, $L(u)$, described in Theorem 1, such that (2) holds.*

We shall divide the proof of this theorem into several stages. Let

$$F_a(s) = \sum_{n=1}^{\infty} \frac{f_a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{f(n)}{n^{s+ia}}.$$

**Lemma 1.** *If the conditions of Theorem 2 are fulfilled, then there exist $a$, $C$, $L(1/(\sigma-1))$ such that*

$$F_a(s) = CL\left(\frac{1}{\sigma-1}\right) |(s-1)|^{\tau} + o\left(\frac{1}{(\sigma-1)^{\tau}}\right) \tag{6}$$

*as $\sigma \to 1+0$, uniformly in any strip $|t| \leq M$.*

We note that in Lemma 1 it is sufficient to require only $\tau > 0$, and not $\tau > 1/2$, as in Theorem 2.

**Proof.** Condition (3), together with condition (5) following from (4), ensures the possibility of representing $F(s)$ in the form

$$F(s) = \exp\left(\sum_p \frac{f(p)}{p^s} + H(s)\right), \tag{7}$$

where $H(s)$ is regular in the region $\sigma > 1/2 + \gamma = 1 - \varepsilon$. Using (7), we obtain for any $t$

$$\frac{|F(\sigma+it)|}{\zeta^{\tau}(\sigma)} = \exp\left\{-\sum_p \frac{\tau - \operatorname{Re} f(p) p^{-it}}{p^{\sigma}} + \operatorname{Re} H(s) + H_1(\sigma)\right\}, \tag{8}$$

where

$$H_1(\sigma) = \tau \sum_p \left(\frac{1}{p^{\sigma}} + \log\left(1 - \frac{1}{p^{\sigma}}\right)\right).$$

Summation by Abel gives

$$\sum_p \frac{\tau - \operatorname{Re} f(p) p^{-it}}{p^{\sigma}} = (\sigma-1) \sum_{n=2}^{\infty} \frac{s(n)}{n_1^{\sigma} \log n_1} + \sum_{n=2}^{\infty} \frac{s(n)}{n_1^{\sigma} \log^2 n_1}, \tag{9}$$

where $n < n_1 < n+1$.

It is easy to show that the limit of the left-hand side of (9) as $\sigma \to 1+0$ exists if and only if there exists

$$\lim_{\sigma\to 1+0} \sum_{n=2}^{\infty} \frac{s(n)}{n_1^{\sigma} \log^2 n_1}.$$

On the other hand, if the latter limit does not exist, then on the basis of condition (1) it is equal to $+\infty$, and, consequently, the limit of the left-hand side in this case is equal to $+\infty$.

Thus we obtain the following alternative:

1) either for all $t$,
$$|F(s)| = o\left(\frac{1}{(\sigma-1)^{\tau}}\right)$$
uniformly for $|t| \leq M$;

2) or there exists an $a$ such that

$$|F_a(s)| = \frac{e^{-\gamma}}{(\sigma-1)^{\tau}} + o\left(\frac{1}{(\sigma-1)^{\tau}}\right), \tag{10}$$

where

$$\gamma = \lim_{\sigma\to 1+0} \sum_{p} \frac{\tau - \operatorname{Re} f_a(p)}{p^{\sigma}} + \operatorname{Re} H(1) + H_1(1).$$

We shall show that in this case, choosing

$$C \sim \exp\left\{-\sum_{p} \frac{\tau - \operatorname{Re} f_a(p)}{p^{\sigma}} + H(1) + H_1(1)\right\},$$

$$L\left(\frac{1}{\sigma-1}\right) = \exp i \sum_{p} \frac{\operatorname{Im} f_a(p)}{p^{\sigma}},$$

we obtain

$$F_a(s) = CL\left(\frac{1}{\sigma-1}\right)\zeta^{\tau}(s) + o\left(\frac{1}{(\sigma-1)^{\tau}}\right) = CL\left(\frac{1}{\sigma-1}\right)\frac{1}{(s-1)^{\tau}} + o\left(\frac{1}{(\sigma-1)^{\tau}}\right)$$

uniformly in $t$ in any fixed strip $|t| \leq M$.

Let us note that in this case there exists

$$\lim_{\sigma\to 1+0}\sum_p \frac{\tau-|f_a(p)|}{p^\sigma}=\gamma_1,$$

for otherwise, as above, from (1) we obtain

$$\lim_{\sigma\to 1+0}\sum_p \frac{\tau-|f_a(p)|}{p^\sigma}=+\infty,$$

$$\frac{|F_a(s)|}{\zeta^\tau(\sigma)}\le \zeta^{-\tau}(\sigma)\sum_{n=1}^{\infty}\frac{|f_a(n)|}{n^\sigma}=\exp\left\{\sum_p \frac{\tau-|f(p)|}{p^\sigma}+H_3(\sigma)\right\}=o(1),$$

which contradicts (10), since $e^{-\gamma}\neq 0$.

Let $K$ be an arbitrary fixed constant and $|t|\le K(\sigma-1)$; then, putting $\arg f_a(p)=\theta_p$, we obtain:

$$\sum_p \frac{\tau-\operatorname{Re} f_a(p)p^{-it}}{p^\sigma}=\gamma_1+\sum_p \frac{|f_a(p)|}{p^\sigma}(1-\cos t\log p)+$$

$$+2\sum_p \frac{|f_a(p)|\sin^2\theta_p}{p^\sigma}\cos t\log p-\sum_p \frac{|f_a(p)|}{p^\sigma}\sin\theta_p\sin t\log p\ge$$

$$\ge c_1\sum_p \frac{|f_a(p)|}{p^\sigma}\sin\theta_p\sin t\log p\ge \frac{\varepsilon_1}{2}\log K, \tag{11}$$

if

$$\sum_p \frac{|f(p)|}{p^\sigma}(1-\cos t\log p)>\varepsilon_1\log\frac{|t|}{\sigma-1}\quad \text{for } |t|\ge K(\sigma-1).$$

The latter is obvious in the case $|f(p)|\ge c>0$.
In the case when condition (4) holds, we proceed as follows: let

$$\alpha=\arccos(1-\varepsilon_2)<2\sqrt{\varepsilon_2};$$

then

$$\sum_p \frac{|f_a(p)|}{p^\sigma}(1-\cos t\log p)\ge \varepsilon_2\sum_p \frac{|f_a(p)|}{p^\sigma}-\varepsilon_2\sum_{\substack{p\\ 1-\varepsilon<\cos t\log p<1}}\frac{|f_a(p)|}{p^\sigma}\ge$$

$$\ge \varepsilon_2\tau\log\frac{1}{\sigma-1}-\gamma_1-\varepsilon_2\sum_{p\le e^{\alpha/t}}\frac{|f_a(p)|}{p^\sigma}-\varepsilon_2\sum_{k=1}^{\infty}\sum_{\exp\frac{2\pi k-\alpha}{t}<p\le\exp\frac{2\pi k+\alpha}{t}}\frac{|f_a(p)|}{p^\sigma}.$$

Since from (1) it follows that

$$\sum_{k=1}^{\infty}\sum_{\exp\frac{2\pi k-\alpha}{t}<p\le\exp\frac{2\pi k+\alpha}{t}}\frac{|f_a(p)|}{p^\sigma}\le t\sum_{k=1}^{\infty}\frac{\exp(1-\sigma)\frac{2\pi k-\alpha}{t}}{2\pi k-\alpha}\times$$

$$\times\sum_{\exp\frac{2\pi k-\alpha}{t}<p\le\exp\frac{2\pi k+\alpha}{t}}\frac{|f_a(p)|\log p}{p^\sigma}\le c_3\sqrt{\varepsilon_2}\log\frac{t}{\sigma-1},$$

on the basis of (4), for $K(\sigma-1)\le|t|\le M$ we have

$$\sum_p\frac{|f(p)|}{p^\sigma}(1-\cos t\log p)\ge\varepsilon_2\tau\log\frac{1}{\sigma-1}-\varepsilon_2\tau\log\frac{1}{t}$$

$$-c_4\varepsilon_2^2\log\frac{t}{\sigma-1}-c_5\ge\varepsilon_3\log K.$$

From (8) follows the validity of (6) for $K(\sigma-1)\le|t|\le M$. In the case $|t|<K(\sigma-1)$, using (1), we obtain

$$\sum_p\frac{\tau-\operatorname{Re}f_a(p)}{p^\sigma}=o(1)\quad\text{as }\sigma\to1+0.$$

The subsequent arguments almost completely coincide with the corresponding arguments in (’ ). This proves Lemma 1.

Next we proceed as follows: let $\sigma_0=1+1/\log x$; then

$$\sum_{n\le x}f_a(n)\log\frac{x}{n}=\frac{1}{2\pi i}\int_{(\sigma_0)}\frac{x^s}{s^2}F_a(s)\,ds.$$

Using (6), we find

$$\frac{1}{2\pi i}\int_{|t|<K(\sigma_0-1)}\frac{x^s}{s^2}F_a(s)\,ds=\frac{CL(\log x)}{\Gamma(\tau)}x\log^{\tau-1}x+o(x\log^{\tau-1}x).$$

Integrating by parts and using Schwarz’ s inequality, we obtain

$$\left|\frac{1}{2\pi i}\int_{|t|\ge K(\sigma-1)}\frac{x^{s-1}}{s^2}F_a(s)\,ds\right|\ll$$

$$\ll\left[\int_{|t|>K(\sigma-1)}\frac{1}{|s|^{3/2}}\left|\frac{F'}{F}(s)\right|^2ds\right]^{1/2}\left[\int_{|t|>K(\sigma-1)}\frac{1}{|s|^{5/2}}|F(s)|^2\,dt\right]^{1/2}.$$

By an almost literal repetition of the corresponding passage from ([1]), taking ([5]) into account, we obtain that

$$\int_{(\sigma_0)} \frac{1}{|s|^{3/2}} \left| \frac{F_a'}{F_a}(s) \right|^2 dt \leq \frac{c_4}{\sigma_0 - 1}.$$

**Lemma 2.** *Let $\alpha > 1$ be such that $\alpha\tau > 1$; then*

$$\int_{(\sigma_0)} \frac{1}{|s|^2} |F(s)|^\alpha \, dt \leq \frac{c_5}{(\sigma_0 - 1)^{\alpha\tau - 1}}.$$

**Proof.** We rely on formula (7) and on the circumstance that

$$\exp\left\{ \frac{\alpha}{2} \sum_p \frac{f(p)}{p^\sigma} \right\} = \sum_{m=1}^{\infty} \frac{\lambda(m)}{m^s}, \quad \text{where} \quad \lambda(p^k) = \frac{f^k(p)}{k!} \left(\frac{\alpha}{2}\right)^k.$$

The latter series converges for $\sigma > 1$, and, see ([2]),

$$\left| \sum_{m \leq x} \lambda(m) \right| \leq c_5 \frac{x}{\log x} \exp\left[ \frac{\alpha}{2} \sum_{p \leq x} \frac{\tau}{p} + o(1) \right] \leq$$

$$\leq c_6 \frac{x}{\log x} \prod_p \left( 1 + \frac{\tau(\alpha/2)}{p} + \frac{\tau^2 (\alpha/2)^2}{2!\, p^2} + \cdots \right) \leq c_7 \sum_{m \leq x} \lambda'(m),$$

where $\lambda'(m)$ are the coefficients of the Dirichlet series.

Applying Parseval' s equality once again and the considerations from ([1]), we obtain the proof of the lemma.

Together with the estimates obtained, this gives

$$\sum_{n \leq x} \frac{f(n)}{n^{ia}} \log \frac{x}{n} = \frac{C}{\Gamma(\tau)} x L(\log x) \log^\tau x + o(x \log^\tau x). \tag{12}$$

The transition from (12) to (2) is carried out by asymptotic differentiation and Abel summation, taking into account that $L(u)$ is a slowly varying function.

Vladimir Pedagogical Institute
named after P. I. Lebedev-Polyanskii

Received
22 XII 1969

## CITED LITERATURE

[1] G. Halasz, *Acta Math. Acad. Sci. Hungarical*, **19**, No. 3-4 (1968).
[2] B. V. Levin, A. S. Fainleib, *DAN*, **188**, No. 3 (1969).

*Note: Figure translations are in progress. See original paper for figures.*