ON THE THEORY OF p-SPACES

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Abstract

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M. M. CHOBAN

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A. V. Arkhangel' skii introduced and studied a new class of topological spaces, the class of p-spaces ([1]). Subsequently, an important role in the development of the theory of p-spaces was played by the works of A. V. Arkhangel' skii, V. I. Ponomarev, V. V. Filippov, the author, and others. In the present article mappings of p-spaces are studied. The work of N. N. Wicke ([6]) is also devoted to this question. All spaces are assumed to be completely regular, and mappings –single-valued and continuous, unless the contrary is explicitly stated. Proofs are omitted throughout.

§ 1. A mapping $f : Z \to X$ of a space Z onto X is called inductively open and bicompact if there exists a subset $Y \subseteq Z$ such that $fY = X$ and the mapping $f|Y$ is open and bicompact.

Theorem 1. *Let $f : Z \to X$ be an inductively open bicompact mapping of a paracompact space Z onto X. Then the space X is weakly paracompact. If, in addition, Z is a p-space, then X is also a p-space.*

A. V. Arkhangel' skii posed the following question: is every weakly paracompact space an open bicompact image of a paracompact space? This question is extremely difficult, and in such a formulation the existence of a solution seems almost incredible. Nevertheless, in a number of cases one can obtain a positive result.

Theorem 2. *Let X be a regular weakly paracompact p-space. Then there exists a paracompact p-space Z which is mapped onto X inductively openly and bicompactly. If, in addition, the space X is complete in the sense of Čech, then Z is also complete in the sense of Čech.*

Theorem 3. *Let X be a weakly paracompact p-space. If the space X satisfies one of the following conditions: a) X is normal; b) X is locally normal; c) X is hereditarily weakly paracompact, then there exists a paracompact p-space Z which is mapped onto X openly and bicompactly. If, in addition, the space X is complete in the sense of Čech, then Z is also complete in the sense of Čech.*

In proving Theorems 2 and 3, an important role is played by a number of special assertions and by the following proposition.

Proposition 1. *Every F_σ-subset of a weakly paracompact space is a weakly paracompact subspace.*

For locally bicompact spaces, generally speaking, a stronger result than Theorem 2 is valid.

Theorem 4. *Every locally bicompact weakly paracompact space X is an open finite-fold image of some locally bicompact paracompact space Z.*

Theorem 1 underlies the proof of the following theorem.

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* A T_1-space X is called a p-space if there exists a countable family $\{\gamma_n \mid n = 1, 2, ...\}$ of open covers in ωX (ωX is the Wallman compactification of the space X) of the space X such that

$$\bigcap_{n=1}^{\infty} \gamma_n x \subseteq X$$

for every point $x \in X$, where

$$\gamma_n x = \bigcup \{U \in \gamma_n \mid x \in U\}.$$

If, in addition, for every point $x \in X$ and every natural number n there exists a natural number m such that $[\gamma_m x]_{\omega X} \subseteq \gamma_n x$, then X is called a strict p-space.

Theorem 5. The product of a countable number of regular weakly paracompact p-spaces is a weakly paracompact space.

Corollary 1. The product of a countable number of weakly paracompact Čech-complete spaces is a weakly paracompact Čech-complete space.

§ 2. **Theorem 6.** Let X be a T_2-space. Then the following conditions are equivalent: a) X is a space of point-countable type*; b) there exist a paracompact p-space Z and an open mapping $f : Z \to X$ such that for every compact set $F \subseteq X$ of countable character in X there exists a compact subset $\Phi \subseteq Z$ for which $f\Phi \supseteq F$.

We note that in ([6]) it was proved that spaces of point-countable type, and only they, are open images of paracompact p-spaces.

Theorem 6 is a strengthening of the theorem from ([6]). This strengthening is of important significance. In particular, it allows one to draw the following conclusion.

Corollary 2. Spaces of countable type, and only they, are open k-covering** images of paracompact p-spaces.

Theorem 7. Every Čech-complete space X is an open image of some paracompact Čech-complete space Z.

§ 3. A mapping $f : X \to Y$ is called regularly complete if there exists a Čech-complete space $\widetilde{X} \supseteq X$ such that for every $y \in Y$ the set $f^{-1}y$ is closed in $\widetilde{X}$.

Examples of regularly complete mappings: 1) compact mappings. 2) Continuous mappings of complete spaces. 3) Let $f : X \to Y$ be a mapping of a metric space (X, ρ) onto Y. If for every point $y \in Y$ the set $f^{-1}y$ is complete with respect to the metric ρ, then the mapping f is regularly complete.

Theorem 8. For every regular topological space X the following conditions are equivalent: a) X is an open regularly complete image of some p-space; b) X is an open regularly complete image of some paracompact p-space Z.

Corollary 3. Every p-space X is an open regularly complete image of some paracompact p-space Z.

It was proved in ([7]) that open regularly complete mappings of a p-space are k-covering.

Theorem 9. Let $f : X \to Y$ be an open regularly complete mapping of a p-space X onto a normal weakly paracompact space Y. Then there exist a paracompact p-space Z and a mapping $g : Z \to X$ of the space Z into X such that: 1) $f(gZ) = Y$; 2) the mapping $\varphi : Z \to Y$, where $\varphi = f \circ g$, is open and compact; 3) the mapping $g : Z \to gZ$ is open and compact.

Corollary 4. Let $f : X \to Y$ be an open regularly complete mapping of a p-space X onto a normal weakly paracompact space Y. Then there exists a weakly paracompact p-space $S \subseteq X$ such that $fS = Y$ and the mapping $f|S$ is open and compact.

Remark 1. In all the theorems indicated it is possible to add the further assertion that the constructed paracompact p-space Z satisfies the following conditions: 1) the weight of the space Z is equal to the weight of the space X; 2) the space Z is perfectly mapped onto some zero-dimensional (in the sense of dim) metric space; 3) $\dim Z \leq \max\{\dim F \mid F$ is a compact subset of the space $X\}$.

* A space X is called a space of point-countable (respectively, countable) type if for every point $x \in X$ (respectively, compact set $F \subseteq X$) there exists a compact set Φ of countable character in X such that $x \in \Phi$ (respectively, $F \subseteq \Phi$).

** A mapping $f : X \to Y$ is called k-covering if for every compact set $F \subseteq Y$ there is a compact set $\Phi \subseteq X$ for which $f\Phi \supseteq F$.

§ 4. A mapping $f : X \to Y$ is called uniformly complete,* if there exists a countable family $\{\gamma_n \mid n = 1, 2, ...\}$ of open covers of the space X such that for every monotone sequence

$$\{F_n \subset \Gamma \in \gamma_n \mid n = 1, 2, ...\}$$

we have

$$\bigcap_{n=1}^{\infty} [F_n] \neq \emptyset,$$

as soon as $F_1 \cap f^{-1}y \neq \emptyset, \dots, F_n \cap f^{-1}y = \emptyset, \dots$ for some point $y \in Y$. We note that every regular complete mapping is uniformly complete.

Theorem 10. Let $\varphi : Y \xrightarrow{\text{onto}} S$ and $\psi : S \xrightarrow{\text{onto}} X$ be open uniformly complete mappings. If Y is a p-space, then there exists a paracompact p-space Z that is mapped openly and regularly completely onto X.

Theorem 11. Let $f : X \to Y$ be an open uniformly complete mapping of a p-space X onto a weakly paracompact space Y. Then Y is a p-space.

Remark 2. Theorem 12 is valid for Wp-spaces.

§ 5. Denote by $A(X)$ the collection of all nonempty subsets of the space X. Put $F(X) = \{L \in A(X) \mid L \text{ is closed in } X\}$. A single-valued mapping $\theta : X \to A(Y)$ is called a set-valued mapping of the space X into Y. In what follows we shall consider only set-valued mappings.

A mapping $\theta : X \to A(Y)$ of a space X into a metric space (Y, ρ) is called a π-Y-mapping if, for every point $x \in X$ and every open set $U \supset \theta^{-1}(\theta x)$ in X, we have

$$\rho(\theta x,\, Y \setminus \theta^{\#} U) > 0,$$

where

$$\theta^{\#} U = \{y \in Y \mid \theta^{-1} y \subset U\}.$$

When the mapping θ^{-1} is single-valued, we obtain the definition of V. I. Ponomarev (see ([4])).

A system $\mathcal{L}$ of subsets of a space X is called an N-system if, for every open set $U \supset L$ in X, where $L \in \mathcal{L}$, there is an open set V in X such that

$$L \subset V \subset [V] \subset U.$$

Theorem 12. Let $\theta : X \to A(Y)$ be a lower semicontinuous** π-Y-mapping of a T_1-space X into a metric space (Y, ρ). Suppose, further, that there exists a space Z such that $X \subset Z \subset \omega X$*** and, for every point $x \in X$, the set $\theta^{-1}\theta x$ is closed in Z. If the family $\{\theta^{-1}\theta x \mid x \in X\}$ is an N-system, then there exists a countable family $\{\gamma_n \mid n = 1, 2, \dots\}$ of open covers in Z of the space X such that

$$\bigcap_{n=1}^{\infty} \gamma_n x \subset X$$

for every point $x \in X$.

A mapping $\theta : X \to A(Y)$ is called XY-bicompact if, for every point $x \in X$, the set $\theta^{-1}(\theta x)$ is bicompact.

Theorem 13. For every regular space X the following conditions are equivalent: 1) X is a strict p-space; 2) there exists an XY-bicompact lower semicontinuous π-Y-mapping $\theta : X \to A(Y)$ into some metric space (Y, ρ); 3) there exists an

XY-bicompact lower semicontinuous π-Y-mapping $\theta : X \to F(Y)$ into some complete zero-dimensional metric space (Y, ρ).

§ 6. Propositions 1 and 2 of ([8]) establish a connection between single-valued and set-valued mappings. It is precisely these propositions that make it possible to translate a number of theorems for single-valued mappings into the language of set-valued mappings.

A mapping $\theta : Y \to A(X)$ is called Y-perfect if the mapping θ is upper semicontinuous and θy is bicompact for every point $y \in Y$.

The theorem from ([6]) is equivalent to the following theorem.

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* Uniformly complete mappings were studied in ([5]).

** A mapping $\theta : X \to A(Y)$ is lower (upper) semicontinuous if, for every open (closed) set A in Y, the set A^{-1} is open (closed) in X, respectively.

*** The space ωX may be replaced by any regular bicompact extension of the space X.

Theorem 14. A T_2-space X is a space of point-countable type if and only if X is an open Y-complete image of some zero-dimensional metrizable space Y.

A mapping $\theta : Y \to A(X)$ is called k-covering if for every bicompactum $\Phi \subset X$ there exists a bicompactum $F \subseteq Y$ such that $\theta F \supset \Phi$.

Corollary 2 makes it possible to draw the following conclusion.

Theorem 15. A T_2-space X is a space of countable type if and only if X is an open Y-complete k-covering image of some metrizable space Y.

A mapping $\theta : Y \to A(X)$ is X-bicompact if $\theta^{-1}x$ is a bicompactum for every point $x \in X$.

From Theorems 3 and 7, respectively, it follows:

Theorem 16. A normal space X is a weakly paracompact p-space if and only if X is an open Y-complete X-bicompact image of some (zero-dimensional) metrizable space Y.

Theorem 17. Every Čech-complete space X is an open Y-complete image of some complete zero-dimensional metrizable space Y.

Mechanical-Mathematical Faculty
of M. V. Lomonosov Moscow State University

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REFERENCES

1. A. V. Arkhangel' skii, Matem. sborn., **67** (109), No. 1, 55 (1965).

2. A. V. Arkhangel' skii, Tr. Moskovsk. matem. obshch., **13**, 3 (1965).

3. A. V. Arkhangel' skii, UMN, **21**, no. 4, 133 (1966).

4. V. I. Ponomarev, Byull. Pol' sk. Akad. nauk, **8**, No. 3, 127 (1960).

5. H. H. Wicke, J. M. Worrell, Duke Math. J., **34**, No. 2, 255 (1967).

6. H. H. Wicke, Proc. Am. Math. Soc., **22**, No. 1, 136 (1969).

7. M. M. Choban, DAN, **190**, No. 2 (1970).

8. M. M. Choban, DAN, **182**, No. 3, 514 (1968).

Note: Figure translations are in progress. See original paper for figures.