ON TWO PROBLEMS FROM GRAPH THEORY

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Abstract

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ON TWO PROBLEMS FROM GRAPH THEORY

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I. The problem of finding a minimal Hamiltonian circuit with weighted arcs

A circuit is called Hamiltonian if it passes through all vertices of a graph, and through each only once (1). Let in a directed graph $G(U)$ with n vertices and set of arcs U, each arc $u_{ij} \in U$, $i, j = 1, 2, \dots, n$, be assigned a weight ρ_{ij}, where the ρ_{ij} are natural numbers from the interval $[1, r]$. For fixed n and r, denote by $\mathfrak{A}_{n,r}$ the set of all such graphs $G(U)$ without loops. It is required, in $G(U)$, to select a Hamiltonian circuit K such that its weight

$$\sum_{u_{ij} \in K} \rho_{ij}$$

is minimal.

If n is not large, then the indicated minimal Hamiltonian circuit K can be found by means of algorithms for solving the traveling-salesman problem (2, 3).* In (3) the algorithm uses memory of $\sim n2^n$ cells and expends $\sim n^2 2^n$ operations; in (2) these quantities are of order, respectively, n and 4^n. In the present paper a sufficiently simple algorithm $\mathcal{A}_\varphi$ is proposed, which, under certain restrictions on r, "almost always" makes it possible to select in $G(U)$ a minimal Hamiltonian circuit. It is also established that "almost all" directed (undirected) graphs contain Hamiltonian circuits (cycles).

We describe the algorithm $\mathcal{A}_\varphi$; by φ denote an arbitrary increasing function of n,

$$\frac{\partial}{\partial n}\varphi > 0, \qquad \lim_{n \to \infty} \varphi = \infty.$$

Assume that in $G(U)$ the vertices are numbered by the numbers $i = 1, 2, \dots, n$; $V^+(V^-)$ denotes the set of even (odd) vertices in $G(U)$; $V_s^\Delta = V^\Delta \setminus V(L_{s-1})$,

$\Delta \in \{+; -\}$, where $V(L_{s-1})$ is the set of vertices belonging to the path $L_{s-1} = [i_1, i_2, \dots, i_s]$ of length $s-1$; $L_0 = i_1$. The algorithm $\mathcal{A}_\varphi$ consists of $n-1$ steps, numbered by $s = 1, 2, \dots, n-1$, and is divided into two stages: 1 and 2.

Stage 1 consists of 2μ steps, where

$$\mu = \left[\frac{n}{2}\left(1 - \frac{1}{\varphi}\right)\right].$$

At the first step $\mathcal{A}_\varphi$ fixes the vertex $i_1 = 1$ and seeks the first, in order, vertex $i_2 \in V^-$, $i_2 \neq i_1$, for which $G(U)$ contains an arc $u_{i_1 i_2}$ such that

$$\rho_{i_1 i_2} = \min_{i \in V_1^-} \rho_{i_1 i};$$

the path $L_1 = u_{i_1 i_2}$ of length 1 is regarded as constructed. Suppose that at step $s-1$ a path L_{s-1} of length $s-1$ has been selected, which begins at vertex $i_1 = 1$ and ends at vertex i_s. If $s \leq \mu - 1$ ($\mu \leq s \leq 2\mu$), then at step s the first, in order, arc $u_{i_s i_{s+1}}$ is chosen such that

$$i_{s+1} \in V_s^- \quad \text{and} \quad \rho_{i_s i_{s+1}} = \min_{i \in V_s^-} \rho_{i_s i}$$

$$\left(i_{s+1} \in V_s^+ \quad \text{and} \quad \rho_{i_s i_{s+1}} = \min_{i \in V_s^+} \rho_{i_s i}\right).$$

After this $u_{i_s i_{s+1}}$ is adjoined to L_{s-1}: the path L_s of length s is constructed, and one proceeds to the $(s+1)$-st step. If for none of the indicated vertices i_{s+1} in $G(U)$ there exists any arc $u_{i_s i_{s+1}}$, then $\mathcal{A}_\varphi$ stops at the s-th step. The result of stage 1 is the path $L_{2\mu}$, which begins at vertex i_1 and ends at vertex $i_{2\mu+1}$.

Stage 2 begins with step $s = 2\mu + 1$, at which $\mathcal{A}_\varphi$ looks for a vertex $i_{2\mu+2} \in (V \setminus V(L_{2\mu}))$ such that there exists a pair of arcs $u_{i_{2\mu+1} i_{2\mu+2}}, u_{i_{2\mu+2} i_1}$ with weight

$$(\rho_{i_{2\mu+1} i_{2\mu+2}} + \rho_{i_{2\mu+2} i_1}) = \min_{i \in (V \setminus V(L_{2\mu}))} (\rho_{i_{2\mu+1} i} + \rho_{i i_1}).$$

The indicated pair of arcs closes $L_{2\mu}$ into a circuit $K_{2\mu+2}$ of length $2\mu + 2$. Suppose that at the $(s-1)$-st step of stage 2, $2\mu + 2 \leq s \leq n-1$, an elementary circuit K_s consisting of s arcs $u_{i_{k_j}k}$, $k = 1, 2, \dots, s$, has been selected, and let i_s be the first vertex in order not belonging to K_s. Then at the s-th step $\mathcal{A}_\varphi$ looks for the first arc $u_{i_{k_j}k} \in K_s$ in order such that the conditions are satisfied: in $G(U)$ there exists a pair of arcs $u_{i_{k_i}s}, u_{i_{s_j}k}$, and the quantity $(\rho_{i_{k_i}s} + \rho_{i_{s_j}k} - \rho_{i_{k_j}k})$ is minimal. After this the pair $u_{i_{k_i}s}, u_{i_{s_j}k}$ is adjoined to the circuit K_s, and the

* The formulated problem can be solved by the well-known "branch and bound" method (⁴), for which, however, there are no estimates.

arc $u_{i_{k_j}k}$ is deleted from K_s: the elementary circuit K_{s+1} is constructed, and one proceeds to the $(s+1)$-st step. At the s-th step of stage 2, $\mathcal{A}_\varphi$ terminates in two cases: either when for the vertex i_s in $G(U)$ no pair of arcs $u_{i_{k_i}s}, u_{i_{s_j}k}$ is found, or when $s = n$, since K_n is a Hamiltonian circuit.

It is not hard to see that for every graph $G(U) \in \mathfrak{A}_{n,r}$ the algorithm $\mathcal{A}_\varphi$ uses on the order of n^2 memory cells and on the order of n^2 operations. All modifications of $\mathcal{A}_\varphi$ considered below ($\mathcal{A}^{(n)}, \mathcal{A}^*_\varphi$ and $\mathcal{A}(W)$) have the same efficiency.

We shall assume that the quantity r, generally speaking, is a function of n: $r = r(n)$; moreover, always $r(n) \geq 1$. We denote the cardinality of any set E by $|E|$; $|\mathfrak{A}_{n,r}| = ([r])^{n^2-n}$. Let a sequence $\mathfrak{A}_{n,r}$, $n = 1, 2, \ldots$, be given, where $r = r(n)$, and let $\mathfrak{A}^{(\omega)}_{n,r}$ denote the set of all graphs $G(U) \in \mathfrak{A}_{n,r}$ possessing some property ω_n. We shall say that almost all graphs $G(U)$ from $\mathfrak{A}_{n,r}$ possess the property ω_n if

$$\lim_{n\to\infty} \frac{|\mathfrak{A}^{(\omega)}_{n,r}|}{|\mathfrak{A}_{n,r}|} = 1.$$

Theorem 1. *For an arbitrary function $r(n)$, $\mathcal{A}_\varphi$ finds a Hamiltonian circuit for almost all graphs $G(U) \in \mathfrak{A}_{n,r}$.*

Corollary 1. *For any $r = r(n)$, almost all graphs from $\mathfrak{A}_{n,r}$ possess Hamiltonian circuits.*

Corollary 2. *Almost all graphs $G(U) \in \mathfrak{A}_{n,r}$ are strongly connected.* Let

$$c_1 = 1/\sqrt{2}, \qquad c_2 = \sqrt{2}.$$

Theorem 2. *If $r \leq c_1\sqrt{n/\ln n}$, then for almost all graphs $G(U) \in \mathfrak{A}_{n,r}$, $\mathcal{A}_\varphi$ finds a minimal Hamiltonian circuit.*

Suppose $\mathcal{A}_\varphi$ selects in $G(U) \in \mathfrak{A}_{n,r}$ some Hamiltonian circuit K_n, whose weight we denote by

$$\sigma_\varphi = \sum_{u_{ij}\in K_n} \rho_{ij}.$$

We denote by σ the weight of a minimal Hamiltonian circuit contained in $G(U)$. If there exists a sequence ε_n, $n = 1, 2, \ldots$, $\varepsilon_n \geq 0$,

$$\lim_{n\to\infty} \varepsilon_n = 0,$$

for which the property $\sigma_\varphi \leq (1+\varepsilon_n)\sigma$ holds for almost all graphs $G(U) \in \mathfrak{A}_{n,r}$, then we shall say that $\mathcal{A}_\varphi$ almost always leads to an asymptotically exact solution. We denote by $\mathcal{A}^{(n)}$ such an algorithm $\mathcal{A}_\varphi$ in which

$$\mu = \left[\frac{n}{2} - \varphi_n\right],$$

where φ_n is an increasing function of n, $\varphi_n \leq \ln\ln n$;

$$\frac{\partial}{\partial n}\varphi_n > 0, \qquad \lim_{n\to\infty}\varphi_n = \infty.$$

Theorem 3. *If*

$$r \leq \frac{n}{\ln n}\,\frac{1}{\varphi_n},$$

then the algorithm $\mathcal{A}^{(n)}$ almost always leads to an asymptotically exact solution.

We shall consider our problem in the following probabilistic formulation. Let, in a directed n-vertex graph G, for each pair of vertices i, j, $i, j = 1, 2, \dots, n$, the arc u_{ij} appear with probability $1-q$, $0 \leq q < 1$, independently of the other arcs.

Theorem 4. *If $q \leq 1 - c_2\sqrt{\ln n/n}$, then with probability $P \geq 1-\delta$, $\delta \to 0$ as $n \to \infty$, the algorithm $\mathcal{A}_\varphi$ selects in G a Hamiltonian cycle.*

Let us agree to denote by G_n such a graph G in which, if an arc u_{ij} is present, then with (conditional) probability p_ν it is assigned weight

$$\rho_{ij} = \nu, \quad \nu = 1, 2, \dots, r; \quad \sum_{\nu=1}^{r} p_\nu = 1.$$

Generally speaking, the quantities q and p_ν, $1 \leq \nu \leq r$, are functions of n: $q = q(n)$ and $p_\nu = p_\nu(n)$. We denote the integral distribution function by

$$F_\nu = \sum_{t=1}^{\nu} p_t; \quad F_0 = 0.$$

We shall call a graph of the form G_n random.

Theorem 5. *If $p_1 \geq c_2\sqrt{\ln n/n}$, then with probability $P \geq 1-\delta_n$, $\lim_{n\to\infty}\delta_n = 0$, the algorithm $\mathcal{A}_\varphi$ selects in G_n a minimum Hamiltonian cycle.*

Introduce the following notation: the parameter α runs through the values

$$2, 3, \dots, 2r; \quad p^*_\alpha = \sum_{\nu=1}^{\alpha-1} p_\nu p_{\alpha-\nu}, \quad \text{if } \alpha \leq r+1, \quad \text{and}$$

$$p^*_\alpha = \sum_{t=0}^{2r-\alpha} p_{\alpha-r+t}\,p_{r-t}, \quad \text{if } r+2 \leq \alpha \leq 2r; \quad \Phi_\alpha = \sum_{\gamma=2}^{\alpha} p^*_\gamma;$$

$$\Phi_1 = 0; \quad Q = 2q(1-q) + q^2;$$

$$f(\varphi; p_1, \dots, p_r) = 2\sum_{\nu=1}^{r-1} \frac{\left(1-(1-q)F_\nu\right)^{n/2\varphi}}{(1-q)F_\nu} + 2\sum_{\alpha=2}^{2r-1} \frac{\left(1-(1-Q)\Phi_\alpha\right)^{\mu}}{(1-Q)\Phi_\alpha} + \sum_{\alpha=2}^{2r-1} \left(1-(1-Q)\Phi_\alpha\right)^{n/2\varphi};$$

$\psi(n)$ is an arbitrary function increasing with n,

$$\frac{\partial}{\partial n}\psi(n) > 0, \quad \lim_{n\to\infty} \psi(n) = \infty.$$

Theorem 6. *If, under the conditions of Theorem 4,*

$$r \le \frac{n}{\ln \psi(n)}$$

and the probability distribution for the weights ρ_{ij} satisfies the inequality

$$f(\varphi; p_1, \dots, p_r) \le n/\psi(n),$$

then the algorithm $\mathcal{A}_\varphi$, applied to the graph G_n, yields, with probability $P \ge 1-\delta_n$, $\lim_{n\to\infty} \delta_n = 0$, an asymptotically exact solution.

The results formulated above are also valid for the case where the weights ρ_{ij} take integer values from the interval $[r_1, r_2]$, where r_1 is bounded below by a positive constant, $r = r_2 - r_1$.

Let $\widetilde{\mathfrak{A}}_{n,r}$ denote the set of all undirected n-vertex graphs $\widetilde{G}(\widetilde{U})$ in which the edges $\tilde{u}_{ij} \in \widetilde{U}$ are assigned weights ρ_{ij}, where ρ_{ij} are natural numbers from $[1, r]$, and let $\widetilde{G}_n$ denote an undirected random n-vertex graph with weighted edges. The algorithm $\mathcal{A}_\varphi$ is suitable for selecting minimum Hamiltonian cycles on $\widetilde{G}(\widetilde{U}) \in \widetilde{\mathfrak{A}}_{n,r}$ and on $\widetilde{G}_n$. In this case all the theorems 1-6 and corollaries 1 and 2 formulated above are valid, with the sole difference that the constants c_1 and c_2 are replaced respectively by the constants $\tilde{c}_1 = 1/2$ and $\tilde{c}_2 = 2$, while the notations $G(U)$,

$\mathfrak{A}_{n,r}$, G_n and the term "contour" are replaced, respectively, by the notations $\widetilde{G}(\widetilde{U})$, $\widetilde{\mathfrak{A}}_{n,r}$, $\widetilde{G}_n$ and the term "cycle."

Let $\mathcal{A}^*_\varphi$ denote the algorithm $\mathcal{A}_\varphi$ that "skips" the $(2\mu+1)$-st step, i.e., does not close the path $L_{2\mu}$ into the contour $K_{2\mu+2}$. Then the algorithm $\mathcal{A}^*_\varphi$ will find a minimal (i.e., having minimum weight) Hamiltonian path on directed graphs with weighted arcs and a minimal Hamiltonian chain on undirected graphs with weighted edges. For minimal Hamiltonian paths and chains, statements analogous to Theorems 1-6 hold.

II. **The problem of a minimum covering of a graph by edges.** A covering of a graph $\widetilde{G}(\widetilde{U})$ is a set of edges $W \subset \widetilde{U}$ such that every vertex of $\widetilde{G}(\widetilde{U})$ is incident to at least one edge from W (1). From the analogue of Corollary 1 for undirected graphs it follows that, for an arbitrary function $r(n)$, the proportion of graphs $\widetilde{G}(\widetilde{U}) \in \widetilde{\mathfrak{A}}_{n,r}$ possessing coverings tends to 1 as n grows.

We shall consider the set $\widetilde{\mathfrak{A}}_{n,1}$ of all n-vertex undirected graphs without loops whose edges have no weights or, equivalently, $\rho_{ij} = 1$ for every $\tilde{u}_{ij} \in \widetilde{U}$. A covering W with the smallest number $|W|$ of edges is called minimal (for $\widetilde{G}(\widetilde{U}) \in \widetilde{\mathfrak{A}}_{n,1}$) and is denoted by $m(\widetilde{G}(\widetilde{U}))$. In (5) an algorithm is given for constructing an asymptotically minimal covering, and it is proved that, for almost all $\widetilde{G}(\widetilde{U}) \in \widetilde{\mathfrak{A}}_{n,1}$,

$$m(\widetilde{G}(\widetilde{U})) \leq \frac{n+1}{2} + \log n.$$

This result can be strengthened.

Theorem 7. For almost all $\widetilde{G}(\widetilde{U}) \in \widetilde{\mathfrak{A}}_{n,1}$, the number of edges in a minimal covering is equal to $[(n+1)/2]$.

Let $\mathcal{A}(W)$ denote the algorithm which first works as $\mathcal{A}_\varphi$, and after a Hamiltonian cycle (contour) has been selected, deletes in it every second (in order) edge (arc).

Theorem 8. For almost all $\widetilde{G}(\widetilde{U}) \in \widetilde{\mathfrak{A}}_{n,1}$, the algorithm $\mathcal{A}(W)$ finds a minimal covering.

Let $\mathfrak{N}$ denote the set of all nonisomorphic n-vertex undirected graphs without loops (the edges are unweighted). In (5), Theorem 4 asserts that, for almost all nonisomorphic graphs $G \in \mathfrak{N}$, the quantity $m(G) \sim n/2$. This result can be strengthened.

Theorem 9. For almost all graphs from $\mathfrak{N}$, the number of edges in a minimal covering is equal to $[(n+1)/2]$.

Theorem 10. Almost all graphs from $\mathfrak{N}$ possess Hamiltonian cycles.

Statements analogous to Theorems 7 and 8 also hold for directed graphs. In (4), p. 42, the so-called weighted covering problem is formulated. The solution of this problem consists in minimizing the sum of the weights of the edges (arcs) entering into the covering, and it can be obtained with the aid of the algorithm $\mathcal{A}(W)$. If the weighted covering problem is considered as applied to graphs with weighted edges (arcs), then statements analogous to Theorems 2–6 are valid for the algorithm.

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1. C. Berge, *Graph Theory and Its Applications*, Moscow, 1962.
2. V. K. Korobkov, R. E. Krichevsky, in: *Mathematical Models and Methods of Optimal Planning*, Novosibirsk, 1966, p. 10.
3. M. Held, R. M. Karp, *Cybernetics Collection*, 9, 202 (1961).
4. A. A. Korbut, Yu. Yu. Finkelstein, *Discrete Programming*, “Nauka,” Moscow, 1969.
5. R. G. Nigmatullin, *Problems of Cybernetics*, 21, 241 (1969).

Note: Figure translations are in progress. See original paper for figures.