INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

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INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

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Let us consider the system of equations

$$\begin{gathered} x_i(t) = \int_a^b Q_i(t, s, x_1(h_1(s)), \dots, x_1(h_{m_1}(s)), \dots, x_n(h_{m_{n-1}+1}(s)), \dots \\ \dots, x_n(h_{m_n}(s)))\, ds + f_i(t), \\ x_i(\xi) = \varphi_i(\xi) \quad \text{for } \xi \in \overline{[a,b]}, \qquad i = 1, \dots, n, \end{gathered} \tag{1}$$

under the following assumptions.

In the domain $t, s \in [a,b]$, $|u_j| < \infty$, $j = 1, \dots, N$ $(N = m_n)$, the function $Q_i(t, s, u_1, \dots, u_N)$ $(i = 1, \dots, n)$ is measurable in s for all $t, u_1, \dots, u_N$ and is continuous with respect to the totality of arguments $u_1, \dots, u_N$ for all t and almost all s. The function

$$\sup_{|u_j| \le \gamma,\ j=1,\dots,N} |Q_i(t, s, u_1, \dots, u_N)|$$

is summable in s for any $\gamma > 0$ and all t. For any $\gamma > 0$ and every t_0,

$$\lim_{t \to t_0} \int_a^b \sup_{|u_j| \le \gamma,\ j=1,\dots,N} \left| Q_i(t_0, s, u_1, \dots, u_N) - Q_i(t, s, u_1, \dots, u_N) \right| ds = 0.$$

On $[a,b]$ the functions $f_1, \dots, f_n$ are continuous, while $h_1, \dots, h_N$ are measurable and bounded. On the set $[\alpha, a] \cup [b, \beta]$, where

$$\alpha = \min\left\{ a, \inf_{s \in [a,b],\ j=1,\dots,N} h_j(s) \right\}, \qquad \beta = \max\left\{ b, \sup_{s \in [a,b],\ j=1,\dots,N} h_j(s) \right\},$$

the functions $\varphi_1, \dots, \varphi_n$ are bounded and may have only a finite number of points of discontinuity.

Denote

$$D_j = \{s \in [a,b] \mid h_j(s) \in [a,b]\}, \qquad E_j = \{s \in [a,b] \mid h_j(s) \in \overline{[a,b]}\},$$

$j = 1, \dots, N$. On the set of n-dimensional vector functions

$$y(t) = \{y_1(t; s, \xi_1, \dots, \xi_k), \dots, y_n(t; s, \xi_1, \dots, \xi_k)\}$$

of the argument $t \in [a,b]$ and parameters $s, \xi_1, \dots, \xi_k$, define the operator $T_h(y)$ as follows: $T_h(y) = y_h$, where $y_h = \{y_{h_1}, \dots, y_{h_N}\}$ is an N-dimensional vector function with components

$$y_{h_j} = \begin{cases} y_i(h_j(t); s, \xi_1, \dots, \xi_k), & \text{for } t \in D_j, \\ 0, & \text{for } t \in E_j, \end{cases} \qquad j = 1, \dots, N,$$

and the index i is determined from the inequality

$$m_{i-1} + 1 \le j \le m_i \qquad (m_0 = 0). \tag{2}$$

By $\varphi^h = \{\varphi^{h_1}, \dots, \varphi^{h_N}\}$ we denote the N-dimensional vector function with components

$$\varphi^{h_j}(t) = \begin{cases} 0, & \text{for } t \in D_j, \\ \varphi_i(h_j(t)), & \text{for } t \in E_j, \end{cases} \qquad j = 1, \dots, N,$$

where the index i is determined by inequalities (2).

Using the introduced notation, we shall write system (1) in the form

$$x(t) = \int_a^b Q(t, s, x_h(s) + \varphi^h(s))\, ds + f(t),$$

where $Q(t,s,u) = \{Q_1(t,s,u_1,\dots,u_N), \dots, Q_n(t,s,u_1,\dots,u_N)\}$, $u = \{u_1, \dots, u_N\}$, $f(t) = \{f_1(t), \dots, f_n(t)\}$.

1. Let $C_{n[a,b]}$ be the space of n-dimensional vector functions $x(t)$ with components $x_i(t)$ continuous on $[a,b]$ and with norm $\|x\| = \max_{t\in[a,b],\, i=1,\dots,n} |x_i(t)|$.

Theorem 1. *The operator*

$$Q(x) = \int_a^b Q(t, s, x_h(s) + \varphi^h(s))\, ds$$

acts in $C_{n[a,b]}$ and is completely continuous.

We shall call a solution of system (1) a fixed point in $C_{n[a,b]}$ of the operator $Q(x)$. Some consequences of Theorem 1 on the existence and uniqueness of a solution of the equation $x = Q(x) + f$ are given without proof in (^{1},^{2}).

To the n-dimensional system (1) we assign the corresponding N-dimensional system

$$z(t) = \int_a^b Q_h(t, s, z(s))\, ds + F(t), \tag{3}$$

where $z(t) = \{z_1(t), \dots , z_N(t)\}$, $F(t) = f_h(t) + \varphi^h(t)$. We consider this system on the set of N-dimensional vector functions with measurable bounded components.

Theorem 2. *System (1) has a solution x if and only if there exists a solution z of system (3). The relations $x(t) = \int_a^b Q(t, s, z(s))\, ds+f(t)$ and $z(t) = x_h(t)+\varphi^h(t)$ establish a one-to-one correspondence between the set of solutions x of system (1) and the set of solutions z of system (3).*

The theorem stated develops the idea of the well-known substitution of A. I. Loginov (^{3}) and makes it possible to reduce a number of problems concerning system (1) to already studied questions concerning system (3) without deviating argument. Thus, assertions on integral inequalities and their consequences as applied to differential equations ($^{4-6}$) extend, by virtue of Theorem 2, to systems with deviating argument. Analogous extensions are obtained by certain approximate and qualitative methods. In other words, assertions about system (3) lead, by virtue of Theorem 2, to the corresponding assertions about system (1).

2. In the case

$$Q_i(t, s, u_1, \dots , u_N) = \lambda \sum_{j=1}^{N} K_{ij}(t, s)u_j$$

we denote

$$K(t, s) = \begin{pmatrix} K_{11}(t, s) \dots K_{1N}(t, s) \\ \dots\dots\dots\dots\dots \\ K_{n1}(t, s) \dots K_{nN}(t, s) \end{pmatrix}, \qquad \Phi(t) = \int_a^b K(t, s)\varphi^h(s)\, ds.$$

Thus, in the linear case system (1) has the form

$$x(t) = \lambda \int_a^b K(t, s)x_h(s)\, ds + f(t) + \lambda\Phi(t). \tag{4}$$

The Fredholm-Riesz theory is applicable to such a system. Therefore, for example, the following is true:

Theorem 3. *If system (4) is uniquely solvable for some pair of vector functions* $\varphi = \{\varphi_1, \dots, \varphi_n\}$, $f = \{f_1, \dots, f_n\}$, *then it is uni-*

is uniquely solvable for arbitrary φ and f. If, for some pair φ and f, system (4) does not have the property of unique solvability, then it will not be uniquely solvable for any φ and f.

Let $P_h(t,s)$ denote the $N \times N$ matrix each column of which is the result of applying the operator T_h to the corresponding column of the $n \times N$ matrix $P(t,s)$. Thus, in the linear case, system (3) has the form

$$z(t) = \lambda \int_a^b K_h(t,s) z(s)\, ds + F(t). \tag{5}$$

It follows from Theorem 2 that the characteristic numbers of system (4) and system (5) coincide.

Below we shall assume that $K_{ij}(t,s)$ $(i = 1, \dots, n;\ j = 1, \dots, N;\ t, s \in [a,b])$ is square summable in s for each t, and that the function

$$K_{ij}(t,s)^2\, ds$$

is continuous on $[a,b]$.

Denote

$$R(t,s;\lambda) = K(t,s) + \lambda \int_a^b K(t,\tau)\Gamma(\tau,s;\lambda)\, d\tau,$$

where $\Gamma(t,s;\lambda)$ is the resolvent of the kernel $K_h(t,s)$ of system (5). Then for the solution $x(t)$ of system (4) we have

$$\begin{aligned} x(t) &= \lambda \int_a^b R(t,s;\lambda)[f_h(s) + \lambda \Phi_h(s)]\, ds + f(t) + \lambda \Phi(t) \\ &= \lambda \int_a^b R(t,s;\lambda)[f_h(s) + \varphi^h(s)]\, ds + f(t). \end{aligned}$$

The resolvent $R(t,s;\lambda)$ of the expressed kernel $K(t,s)$ can be constructed on the basis of Theorem 2 in finite form. For the case $n = 1$, $N = 1$ such a resolvent is given in (7).

3. The system of differential equations

$$x_i'(t) + \sum_{r=1}^{n} a_{ir}(t)x_r(t) = \sum_{k=1}^{n} \sum_{j=m_{k-1}+1}^{m_k} b_{ij}(t)x_k(h_j(t)) + f_i(t), \qquad t \in [a,b],$$

$$x_i(\xi) = \varphi_i(\xi) \quad \text{for } \xi \notin [a,b], \qquad i = 1, \dots, n,$$

has, in the notation adopted above, the form

$$x'(t) + A(t)x(t) = B(t)[x_h(t) + \varphi^h(t)] + f(t), \tag{6}$$

where

$$A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \cdot & \cdot & \cdot \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \qquad B(t) = \begin{pmatrix} b_{11}(t) & \cdots & b_{1N}(t) \\ \cdot & \cdot & \cdot \\ b_{n1}(t) & \cdots & b_{nN}(t) \end{pmatrix}.$$

By a solution of system (6) we shall mean a vector-function with absolutely continuous components satisfying (6) almost everywhere on $[a,b]$.

Assuming f_i, a_{ir}, b_{ij} $(i, r = 1, \dots, n,\ j = 1, \dots, N)$ to be square summable on $[a,b]$, consider the boundary-value problem

$$x'(t) + A(t)x(t) = B(t)[x_h(t) + \varphi^h(t)] + f(t), \qquad l[x] = c. \tag{7}$$

Let $l[x] = \{l_1[x], \dots, l_n[x]\}$, where $l_1[x], \dots, l_n[x]$ are such linear functionals in $C_n[a,b]$ that the problem $u' + Au = f,\ l[u] = 0$ has, for every f, the solution

$$u(t) = \int_a^b G(t,s)f(s)\,ds,$$

and the matrix $G(t,s)$ $(t, s \in [a,b])$ is square summable in s for each t.

Theorem 4. Problem (7) is uniquely solvable if and only if $x \equiv 0$ is the unique solution of the homogeneous problem

$$x'(t) + A(t)x(t) = B(t)x_h(t), \qquad l[x] = 0.$$

In the case of unique solvability there exists a Green matrix $\mathfrak{G}(t,s)$ of problem (7), that is, for arbitrary f, φ, and c, the solution $x(t)$ of this problem has the form

$$x(t) = \int_a^b \mathfrak{G}(t,s)\,[f(s) + B(s)\varphi^h(s)]\,ds + d(t).$$

Here $d(t)$ is the solution of the problem $u' + Au = 0,\ l[u] = c$.

Denoting by $R(t,s)$ the resolvent of the equation

$$x(t) = \int_a^b G(t,s)B(s)x_h(s)\,ds,$$

we have:

$$\mathfrak{G}(t,s) = G(t,s) + \int_a^b R(t,\tau)G_h(\tau,s)\,d\tau.$$

Corollary. Let $l[x] \equiv x(a),\ h_j(t) \le t,\ j = 1,\dots,N,\ t \in [a,b]$. Then problem (7) is uniquely solvable. If on $[a,b]$ $a_{ir} \ge 0$ $(i,r = 1,\dots,n;\ i \ne r)$, $b_{ij} \ge 0$ $(i = 1,\dots,n,\ j = 1,\dots,N)$, then $\mathfrak{G}(t,s) \ge 0$ in the square $t,s \in [a,b]$.

Let us note that if, for some $l[x]$, problem (7) is uniquely solvable, then there exists an n-dimensional fundamental system of solutions $x^1,\dots,x^n$ of the homogeneous equation $x'(t) + A(t)x(t) = B(t)x_h(t)$. Thus every solution $x(t)$ of equation (6) has the form

$$x(t) = \sum_{i=1}^{n} c_i x^i(t) + \int_a^b \mathfrak{G}(t,s)[f(s) + B(s)\varphi^h(s)]\,ds.$$

4. Using estimates of the spectral radius of the integral equation equivalent to the boundary-value problem, one can obtain solvability conditions expressed in terms of the coefficients of the equation. For example, for the scalar equation

$$x^{(n)}(t) + \sum_{k=1}^{r+1} \sum_{j=m_{k-1}+1}^{m_k} p_j(t)x^{(k-1)}(h_j(t)) = f(t), \qquad t \in [a,b],$$

$$x^{(i)}(\xi) = \varphi_i(\xi) \quad \text{for } \xi \notin [a,b], \quad i = 0,1,\dots,r, \quad r \le n-1, \tag{8}$$

the following is true.

Theorem 5. Let

$$\int_a^b \sum_{k=1}^{r} \sum_{j=m_{k-1}+1}^{m_k} \frac{(h_j(s)-a)^{n-k}(b-s)^{n-r-1}}{(n-k)!}\,\sigma_j(s)|p_j(s)|\,ds < (b-a)^{n-r-1},$$

where

$$\sigma_j(s) = \begin{cases} 1, & \text{if } s \in D_j, \\ 0, & \text{if } s \in E_j, \end{cases} \qquad j = 1, \dots, m_{r+1}.$$

Then equation (8) with boundary conditions $x^{(i)}(a) = 0$, $i = 0, 1, \dots, n-2$, $x^{(r)}(b) = 0$ has a unique solution. If, moreover, $p_j(t) \geq 0$, $j = 1, \dots, m_{r+1}$, $t \in [a, b]$, then the Green function of the problem is negative for $t, s \in [a, b]$.

Analogous solvability criteria for equation (8) with other boundary conditions are given in (^{1}, ^{2}).

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Note: Figure translations are in progress. See original paper for figures.

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