ON THE COMPLEXITY OF DECIDING RECURSIVELY ENUMERABLE SETS

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ON THE COMPLEXITY OF DECIDING RECURSIVELY ENUMERABLE SETS

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The paper considers questions connected with estimates of the complexity of algorithms recognizing membership of natural numbers from a certain finite set in a recursively enumerable set. We use the terminology and concepts introduced in papers ($^{1-3}$).

Let $\mathfrak{M}$ be a recursively enumerable set, and let n be a natural number. We shall call a Φ-algorithm $\mathfrak{A}$ an $(\mathfrak{M}, n)$-deciding one if it is applicable to every natural number x not exceeding n, and for every such x

$$\mathfrak{A}(x) \simeq \Lambda \equiv x \in \mathfrak{M}.$$

We shall call a function f a **lower estimate of the complexity of deciding** the recursively enumerable set $\mathfrak{M}$ if, whatever the natural number n, every $(\mathfrak{M}, n)$-deciding Φ-algorithm has complexity not less than $f(n)$.

We shall call a recursively enumerable set $\mathfrak{M}$ **effectively nonrecursive** if there exists an unbounded general recursive function that is a lower estimate of the complexity of deciding the set $\mathfrak{M}$.

1. Let $\mathfrak{M}$ be a recursively enumerable set. We shall call a general recursive function g an **enumeration** of the set $\mathfrak{M}$ if, for every natural number x, the equivalence

$$x \in \mathfrak{M} \equiv \exists n\,(g(n) = x)$$

holds.

We shall call a general recursive function f **regular** if, for any natural number k, one can indicate a natural number m such that every natural number n, as soon as $f(n) = f(k)$, does not exceed m.

Let f, g be a pair of general recursive functions. By the symbol $f \lhd g$ we shall denote the following assertion: "the function f is nondecreasing and $\forall n(f(n) \leq g(n))$."

1.1. The following theorem is proved comparatively simply.

Theorem 1. a) Let the general recursive function g be an enumeration of the recursively enumerable set $\mathfrak{M}$. Let there exist an unbounded general recursive function f such that $f \lhd g$. Then the set $\mathfrak{M}$ is recursive and the function g is regular.

b) Let the general recursive function g be regular and be an enumeration of the recursively enumerable set $\mathfrak{M}$. Then one can indicate an unbounded general recursive function f such that $f \lhd g$.

Thus, if a regular function g is an enumeration of a recursively enumerable set $\mathfrak{M}$, then a necessary and sufficient condition for the nonrecursiveness of $\mathfrak{M}$ is the absence of an unbounded general recursive function f such that $f \lhd g$.

1.2. The following theorems indicate conditions satisfied by enumerations of effectively nonrecursive sets.

Theorem 2. Let the general recursive function g be an enumeration of an effectively nonrecursive set $\mathfrak{M}$. Then for any partial-recursive-

function φ one can specify a natural number m such that, if φ is general recursive and $\varphi \prec g$, then $\forall n(\varphi(n) \leq m)$.

We introduce the following notation. Let f be a general recursive function. Denote by the symbol $\chi(f)$ the formula

$$\exists m \forall n(f(n) = 0 \vee f(n) = m).$$

Let g be a general recursive function. By the symbol $\vartheta(g)$ we shall denote the following statement: "there exists a partial recursive function φ such that, for every Gödel number k of a general recursive function f such that $f \prec g$ and $\chi(f)$, $\varphi(k)$ is defined and $\forall n(f(n) \leq \varphi(k))$."

Theorem 3. Let the general recursive function g be a listing of a recursively enumerable set $\mathfrak{M}$ such that $\vartheta(g)$. Let h be an unbounded general recursive function. Then one can specify general recursive functions f and d such that: 1) the function f is a listing of the set $\mathfrak{M}$; 2) $\forall n(f(n) \leq d(h(n)))$; 3) the function f is regular if and only if the functions g and h are regular simultaneously.

Corollary 1. Let a regular general recursive function g be a listing of a recursively enumerable set $\mathfrak{M}$ such that $\vartheta(g)$. Then the set $\mathfrak{M}$ is effectively nonrecursive.

1.3. The regularity condition for the listing in Corollary 1 is essential. This is asserted by the following theorem.

Theorem 4. For every infinite recursively enumerable set $\mathfrak{M}$ one can specify a general recursive function g such that: 1) the function g is a listing of the set $\mathfrak{M}$, and for every n the equation $g(x) = n$ has no more than two roots; 2) for every partial recursive function φ one can specify a natural number m such that, if φ is general recursive and $\varphi \prec g$, then $\forall n(\varphi(n) \leq m)$.

2. In what follows we use the terminology and notation of the article (2).

A word R in the alphabet Φ will be called an n-segment of the set $\mathfrak{M}$ if: 1) the word R is an $(n+1)$-dimensional Boolean vector; 2) $\forall m(m \leq n \supset (\sigma_{m+1}(R) = 1 \equiv m \in \mathfrak{M}))$.

Let f be a general recursive function. We shall call a set $\mathfrak{N}$ an f-image of the set $\mathfrak{M}$ if there exists a Φ-algorithm $\mathfrak{A}$ such that, whatever the natural number n, the algorithm $\mathfrak{A}$ is applicable to any Boolean vector R that is an $f(n)$-segment of the set $\mathfrak{M}$, and for every such R the word $\mathfrak{A}(R)$ is an n-segment of the set $\mathfrak{N}$.

By the letter E we denote the general recursive function defined by the equality

$$E(n) = n.$$

Theorem 5. For every recursively enumerable set $\mathfrak{M}$ one can specify a recursively enumerable set $\mathfrak{N}$ such that: 1) whatever the natural number n, in the intersection of the set $\{0, \dots, 3n\}$ and the set $\mathfrak{N}$ there are no more than $[n/8]$ elements; 2) the set $\mathfrak{N}$ is an E-image of the set $\mathfrak{M}$; 3) if the set $\mathfrak{M}$ is nonrecursive, then the set $\mathfrak{N}$ is simple.

Corollary 2. There exist simple non-hypersimple* sets that are not effectively nonrecursive.

A recursively enumerable set $\mathfrak{M}$ will be called **effectively simple** if its complement is not finite and, for every infinite recursively enumerable subset $\mathfrak{N}$ of the complement of $\mathfrak{M}$, one can indicate a natural number n such that $\mathfrak{N}$ contains no more than n numbers (cf. (4)).

The following theorem is comparatively easy to prove.

Theorem 6. *Every effectively simple non-hyperimmune set is effectively nonrecursive.*

Corollary 3. *Every effectively simple set is simple.*

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* We call a recursively enumerable set $\mathfrak{M}$ **non-hypersimple** if there exists a strictly increasing general recursive function f such that, for every natural number n, it is false that the set $\{f(n), f(n)+1, \dots, f(n+1)\}$ is contained in the set $\mathfrak{M}$.

The condition of non-hyperimmunity in Theorem 6 is essential, since there exist effectively simple hyperimmune sets (according to Theorem 4 of paper (^{5}), all such sets are not effectively nonrecursive).

According to Theorem 6, Corollary 2 gives us an example of a simple set that is not effectively simple. The first example of such a set was constructed by J. Sacks (^{6}), but his construction and proof are more complicated.

Corollary 4. *Among simple non-hyperimmune sets that are not effectively simple, one can exhibit both effectively nonrecursive sets and sets that are not effectively nonrecursive.*

Corollary 2, Theorem 6, and Corollary 4 refine Theorem 5 of paper (^{5}).

Let F denote the general recursive function defined by the equality

$$F(n) = 3n.$$

Using Theorem 5, one can prove the following theorem.

Theorem 7. *For any recursively enumerable set $\mathfrak{M}$, one can indicate a recursively enumerable set $\mathfrak{N}$ such that: 1) for every positive integer n, it is false that the set $\{2n, 2n+1, \dots, 3n\}$ is contained in $\mathfrak{N}$; 2) the set $\mathfrak{N}$ is an E-image of the set $\mathfrak{M}$; 3) the set $\mathfrak{M}$ is an F-image of the set $\mathfrak{N}$; 4) if the set $\mathfrak{M}$ is nonrecursive, then the set $\mathfrak{N}$ is simple.*

Theorem 7 makes it possible to obtain theorems on simple non-hyperimmune sets analogous to Theorem 1 of paper (^{5}), Theorems 4.1 and 4.2 of paper (^{7}), etc.

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Note: Figure translations are in progress. See original paper for figures.

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