ON SINGULAR INTEGRAL OPERATORS IN A WEIGHTED HÖLDER SPACE

1970

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Abstract

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UDC 517.9

MATHEMATICS

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ON SINGULAR INTEGRAL OPERATORS IN A WEIGHTED HÖLDER SPACE

(Presented by Academician N. I. Muskhelishvili on 22 VII 1969)

In the present note we set forth some additions to the well-known results of N. I. Muskhelishvili (^{1}) on singular integral equations with piecewise-Hölder coefficients. In the proofs of almost all the theorems given below, we use methods proposed by I. Ts. Gohberg and N. Ya. Krupnik in ($^{2-4}$).

1. Let Γ be a simple smooth closed oriented curve in the plane, enclosing the point $z = 0$. Let $c_1, \dots, c_n$ be some points on Γ, and let μ be a real number, $0 < \mu \leqslant 1$.

By $H_\mu(\Gamma)$ we shall denote the Banach space of functions defined on Γ, satisfying the Hölder condition with exponent μ and with norm

$$\|\varphi\|_H = \sup \frac{|\varphi(t_2) - \varphi(t_1)|}{|t_2 - t_1|^\mu} + \max|\varphi(t)| \qquad (t, t_1, t_2 \in \Gamma,\ t_1 \neq t_2).$$

By $H^0_\mu(\Gamma, c, \dots, c_n)$ we shall denote the subspace of the space $H_\mu(\Gamma)$ consisting of all functions $\varphi(t)$ for which $\varphi(c_j) = 0$ $(j = 1, \dots, n)$.

Let $\alpha_1, \dots, \alpha_n$ be some complex numbers. Consider the function

$$\rho(t) = \prod_{k=1}^{n} (t - c_k)^{\alpha_k}. \qquad (1)$$

By $H^0_\mu(\Gamma, \rho)$ we shall denote the set of all functions $\varphi(t)$ satisfying the condition $\varphi(t) \cdot \rho(t) \in H^0_\mu(\Gamma, c_1, \dots, c_n)$. It is easy to see that the class $H^0_\mu(\Gamma, \rho)$, endowed with the norm $\|\varphi\|_\rho = \|\varphi \cdot \rho\|_H$, is a Banach space.

Using known methods for estimating singular operators (see (^{1}), § 25), one can show that if the numbers $\mu, \alpha_1, \dots, \alpha_n$ satisfy the conditions

$$0 < \mu < 1, \qquad \mu < \operatorname{Re}\alpha_k < \mu + 1 \quad (k = 1, \dots, n), \qquad (2)$$

then the singular integration operator S

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau \qquad (t \in \Gamma)$$

is a linear bounded operator in the space $H^0_\mu(\Gamma, \rho)$.

The following lemma plays an essential role in what follows.*

Lemma. If $a(t) \in H_\mu(\Gamma)$, and the numbers $\mu, \alpha_1, \dots, \alpha_n$ satisfy the relations (2), then the operator $T = aS - SaI$ (I is the identity operator) is a completely continuous operator in the space $H^0_\mu(\Gamma, \rho)$, where $\rho(t)$ is defined by equality (1).

Theorem 1. Let $c(t)$ and $d(t) \in H_\mu(\Gamma)$; the numbers $\mu, \alpha_1, \dots, \alpha_n$ satisfy the relations (2), and $\rho(t)$ is defined by equality (1).

In order that the operator

$$A = cI + dS \qquad (A = cI + SdI)$$

be a Φ_+- or Φ_--operator** in the space $H^0_\mu(\Gamma, \rho)$, it is necessary and sufficient— sufficient that the conditions $c(t) + d(t) \neq 0,\ c(t) - d(t) \neq 0\ (t \in \Gamma)$ be satisfied. If these conditions are satisfied and

$$\chi = (1/2\pi)[\arg((c + d)/(c - d))]_\Gamma,$$

then:

1) for $\chi > 0$ the operator A is left invertible in $H^0_\mu(\Gamma, \rho)$ and $\dim \operatorname{coker} A = \chi$;
2) for $\chi < 0$ the operator A is right invertible in $H^0_\mu(\Gamma, \rho)$ and $\dim \ker A = -\chi$;
3) for $\chi = 0$ the operator A is invertible in $H^0_\mu(\Gamma, \rho)$.

For the case of the Hölder space without weight, Theorem 1 was established in (7).

We note that, in order to establish the theorem formulated above, a new method was needed for proving the necessity of its conditions.

From Theorem 1, in particular, it follows that if the functions $c(t)$ and $d(t) \in H_\mu(\Gamma)$, then the spectrum of the operator A does not depend on the choice of the numbers $\alpha_1, \dots, \alpha_n$ satisfying relations (2). This assertion ceases to be true if the coefficients $c(t)$ and $d(t)$ have discontinuities of the first kind at certain fixed points on Γ (see Theorem 2).

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* For Hölder spaces without weight this lemma was established in (^{5}), § 6.
** For the definition of $\Phi_\pm$- and Φ-operators, see (^{6}).

2. Let x and ξ be two points of the complex plane; μ real, and α a complex number, with $0<\mu<1$, $\mu<\operatorname{Re}\alpha<\mu+1$. By $\nu_{\mu,\alpha}(z,\xi)$ we denote an arc of a circle joining the points z and ξ and possessing the following properties: (I) in the case $\operatorname{Re}\alpha-\mu<1/2$, from the interior points of the arc $\nu_{\mu,\alpha}(z,\xi)$ the segment joining the points z and ξ is seen under the angle $\theta=(\operatorname{Re}\alpha-\mu)2\pi$, and the direction from z to ξ along the arc $\nu_{\mu,\alpha}(z,\xi)$ is counterclockwise; (II) in the case $\operatorname{Re}\alpha-\mu>1/2$, put

$$\nu_{\mu,\alpha}(z,\xi)=\nu_{\mu,\beta}(\xi,z),$$

where $\beta=1+2\mu-\alpha$; (III) in the case $\operatorname{Re}\alpha-\mu=1/2$, by $\nu_{\mu,\alpha}(z,\xi)$ we denote the segment joining the points z and ξ.

By $H_\mu(\Gamma,c_1,\dots,c_n)$ we denote the set of functions continuous from the left on Γ, satisfying everywhere the Hölder condition with exponent μ, except, possibly, at the points $c_1,\dots,c_n$, at which they may have discontinuities of the first kind.

Let $a(t)\in H_\mu(\Gamma,c_1,\dots,c_n)$, and let $\omega=(\mu,\alpha_1,\dots,\alpha_n)$ denote a vector with coordinates satisfying relations (2). To the function $a(t)$ and the vector ω we associate a continuous, closed, naturally oriented curve $V_\omega(a)$, obtained by adding n arcs $\nu_{\mu,\alpha_k}(a(c_k),a(c_k+0))^*$ to the set of values of the function $a(t)$. The function $a(t)$ will be called ω-nonexceptional if $0\notin V_\omega(a)$. The **index** (or ω-**index**) of an ω-nonexceptional function is the number of turns of the curve $V_\omega(a)$ about the point $z=0$:

$$\operatorname{ind}_\omega a=\frac{1}{2\pi}[\arg V_\omega(a)]_\Gamma.$$

Theorem 2. Let $c(t)$ and $d(t)\in H_\mu(\Gamma,c_1,\dots,c_n)$; let the vector

$$\omega=(\mu,\alpha_1,\dots,\alpha_n)$$

satisfy relations (2), and let

$$\rho(t)=\prod_{k=1}^{n}(t-c_k)^{\alpha_k}.$$

In order that the operator

$$A=cI+dS\quad(A=cI+SdI)\tag{3}$$

be a Φ_+- or Φ_--operator in the space $H^0_\mu(\Gamma,\rho)$, it is necessary and sufficient that the following two conditions be satisfied:

I. $\inf|b(t)|>0$ $(t\in\Gamma)$.

II. The function $a(t)/b(t)$ is ω-nonexceptional, where $a(t)=c(t)+d(t)$ and $b(t)=c(t)-d(t)$.

If conditions I and II are satisfied and $\chi=\operatorname{ind}_\omega[a/b]$, then:

1) for $\chi>0$ the operator A is left invertible in $H^0_\mu(\Gamma,\rho)$ and $\dim\operatorname{coker}A=\chi$;

2) for $\chi < 0$ the operator A is right invertible in $H_\mu^0(\Gamma, \rho)$ and $\dim \ker A = -\chi$;

3) for $\chi = 0$ the operator A is invertible in $H_\mu^0(\Gamma, \rho)$.

* The orientation of the curve $V_\omega(a)$ is chosen so that, on the intervals of continuity of the function $a(t)$, the motion along the curve $V_\omega(a)$ is determined by the motion of the variable t on Γ in the positive direction, and along the arcs $\nu_{\mu,\alpha_k}(a(c_k), a(c_k+0))$—from the point $a(c_k)$ to the point $a(c_k+0)$.

It is not difficult to verify that conditions I and II are equivalent to the following:

I′. $\inf |a(t)| > 0$ and $\inf |b(t)| > 0$ $(t \in \Gamma)$.

II′. $\beta_k \neq \operatorname{Re} \alpha_k - \mu$, where

$$\beta_k = (1/2\pi) \arg [a(c_k) b(c_k+0)/a(c_k+0) b(c_k)].$$

Let conditions I and II be fulfilled. Consider the function

$$\psi(t) = \prod_{k=1}^{n} t^{\gamma_k},$$

where the numbers

$$\gamma_k = (1/2\pi) \ln[a(c_k) b(c_k+0)/a(c_k+0) b(c_k)]$$

are chosen so that

$$\operatorname{Re} \alpha_k - \mu > \operatorname{Re} \gamma_k > \operatorname{Re} \alpha_k - \mu - 1.$$

The latter is possible by virtue of condition II′. We also agree that the point of discontinuity for the function t^{γ_k} (if γ_k is not an integer) is the point c_k. It is easy to see that the function

$$g(t) = a(t)/b(t)\psi(t)$$

belongs to the class $H_\mu(\Gamma)$ and $g(t) \neq 0$ $(t \in \Gamma)$. If

$$g_\pm(t) = \exp\left[\tfrac{1}{2}((I \pm S) t^{-\varkappa} g)(t)\right],$$

where

$$\varkappa = (1/2\pi)[\arg g(t)]_\Gamma,$$

then one of the inverse operators to the operator $A = cI + dS$ (the left inverse for $\varkappa \geq 0$, the right inverse for $\varkappa \leq 0$) can be written in the form

$$A^{-1} = g_+^{-1}(t) \left[\frac{1+g(t)}{2} I + \frac{1-g(t)}{2} S \right] \times$$

$$\times \psi_+^{-1}(t) \left[\frac{t^{-\varkappa} + \psi(t)}{2} I + \frac{t^{-\varkappa} - \psi(t)}{2} S \right] b^{-1}(t) g_-^{-1}(t) \psi_-^{-1}(t) I,$$

where

$$\psi_+(t) = \prod_{k=1}^{n} (t - c_k)^{\alpha_k}, \qquad \psi_-(t) = \prod_{k=1}^{n} \left(\frac{t}{t - c_k} \right)^{\alpha_k}.$$

Theorem 2, in its sufficient part, is a refinement of certain results of N. I. Muskhelishvili (see (^{1}), § 97). From Theorem 2 the following theorem on the spectrum of the operator A, defined by equality (3), is easily derived.

Theorem 3. Let $c(t)$ and $d(t) \in H_\mu(\Gamma, c_1, \dots, c_n)$, and let the vector

$$\omega = (\mu, \alpha_1, \dots, \alpha_n)$$

satisfy relations (2), and

$$\rho(t) = \prod_{k=1}^{n} (t - c_k)^{\alpha_k}.$$

Then the complement $C\Phi_A$ to the Φ-set* of the operator A in the space $H_\mu^0(\Gamma, \rho)$ consists of the union of the set of values of the functions $a(t)$ and $b(t)$ and the set of complex numbers λ satisfying, for at least one pair of numbers k and x ($k = 1, \dots, n$; $0 \le x \le 1$), the equation

$$[a(c_k) - \lambda] \frac{\sin \theta_k (1 - x)}{\sin \theta_k} e^{ix\theta_k} + [a(c_k + 0) - \lambda] \frac{\sin \theta_k x}{\sin \theta_k} e^{i\theta_k (x-1)} = 0,$$

where

$$\theta_k = \pi - 2\pi(\operatorname{Re} \alpha_k - \mu).$$

The spectrum of the operator A in $H_\mu^0(\Gamma, \rho)$ consists of all points of $C\Phi_A$ and the points

$$\lambda \in \Phi_A,$$

for which

$$\operatorname{ind}_\omega [(a - \lambda)/(b - \lambda)] \neq 0.$$

3. By $H_\mu^m(\Gamma, \rho)$ we denote the space of vector-functions

$$\Psi = (\psi_1, \dots, \psi_n)$$

with components $\psi_j \in H_\mu^0(\Gamma, \rho)$. Consider in the space $H_\mu^m(\Gamma, \rho)$ the matrix singular integral operators

$$(K\Psi)(t) = \mathcal{G}(t)\Psi(t) + \mathcal{D}(t)(S\Psi)(t),$$

$$(L\Psi)(t) = \mathcal{G}(t)\Psi(t) + (S\mathcal{D}\Psi)(t),$$

where $\mathcal{G}(t)$ and $\mathcal{D}(t)$ are matrix-functions of order m, each element of which belongs to the set $H_\mu^m(\Gamma, c_1, \dots, c_n)$ (i.e. $\mathcal{G}(t)$ and $\mathcal{D}(t) \in H_\mu^m(\Gamma, c_1, \dots, c_n)$), and S is the matrix operator of singular integration.

If

$$\mathcal{G}(t) = \|g_{ij}\|_{i,j=1}^{m},$$

then we associate with it the discontinuous matrix-function

$$V_\omega(\mathcal{G}) = \|V_\omega(g_{ij})\|_{i,j=1}^{m},$$

where $\omega = (\mu, \alpha_1, \dots, \alpha_n)$ is a vector whose coordinates satisfy relations (2).

* The Φ-set of the operator A is the set of those points λ of the complex plane for which the operator $A - \lambda I$ is a Φ-operator.

We shall call a matrix-function $\mathcal{G}(t) \in H_\mu^m(\Gamma, c_1, \dots, c_n)$ ω-nonsingular if the function $\det V_\omega(\mathcal{G})$ does not vanish.

Theorem 4. Let $\mathcal{G}(t)$ and $\mathcal{D}(t) \in H_\mu^m(\Gamma, c_1, \dots, c_n)$. In order that the operator $K = \mathcal{G}I + \mathcal{D}S$ $(L = \mathcal{G}I + S\mathcal{D}I)$ be a Φ-operator in the space $H_\mu^m(\Gamma, \rho)$, it is necessary and sufficient that the following two conditions be fulfilled:

I. $\inf|\det \mathcal{B}(t)| > 0$ $(t \in \Gamma)$.

II. The matrix $\mathcal{B}^{-1}\mathcal{A}$ $(\mathcal{A}\mathcal{B}^{-1})$ is ω-nonsingular, where $\mathcal{A}(t) = \mathcal{G}(t) + \mathcal{D}(t)$ and $\mathcal{B}(t) = \mathcal{G}(t) - \mathcal{D}(t)$.

If conditions I and II are fulfilled, then

$$\operatorname{ind} K = \operatorname{ind}\det V_\omega(\mathcal{B}^{-1}\mathcal{A}), \qquad (\operatorname{ind} L = \operatorname{ind}\det V_\omega(\mathcal{A}\mathcal{B}^{-1}))^{*}.$$

It is easily verified that conditions I and II are equivalent to the following:

I′. $\inf|\det \mathcal{A}(t)| > 0$, $\inf|\det \mathcal{B}(t)| > 0$ $(t \in \Gamma)$.

II′. For each eigenvalue $\lambda_j^{(k)}$ of the matrix

$$\mathcal{A}^{-1}(c_k + 0)\mathcal{B}(c_k + 0)\mathcal{B}^{-1}(c_k)\mathcal{A}(c_k)$$

$$(\mathcal{B}(c_k + 0)\mathcal{A}^{-1}(c_k + 0)\mathcal{A}(c_k)\mathcal{B}^{-1}(c_k))$$

the relation

$$\operatorname{Re}\alpha_k \neq \mu + \beta_j^{(k)} \qquad (j = 1, \dots, m;\ k = 1, \dots, n),$$

holds, where $\beta_j^{(k)} = (1/2\pi)\arg\lambda_j^{(k)}$, $0 \leq \beta_j^{(k)} < 1$.

4. All the results stated admit a generalization to the case when Γ consists of a finite number of closed smooth contours, and to the case when Γ consists of a finite number of smooth simple closed and open arcs. In doing so, the method described in (^{2}) is used.

Let us illustrate the results of the second point for the case when $A = S$ (i.e. $c(t) = 0$ and $d(t) = 1$) and Γ is a contour consisting of p simple smooth oriented open arcs. Denote by $c_1, \dots, c_{2p}$ the ends of these arcs, and by $\rho(t)$ the weight defined by the equality

$$\rho(t) = \prod_{k=1}^{2p} (t - c_k)^{\alpha_k},$$

where $0 < \mu < 1,\ \mu < \operatorname{Re} \alpha_k < \mu + 1\ (k = 1, \dots, 2p)$.

Theorem 5. In order that the operator S be a Φ_+- or Φ_--operator in the space $H^0_\mu(\Gamma, \rho)$, it is necessary and sufficient that $\operatorname{Re} \alpha_k - \mu \neq \frac{1}{2}$ for all $k = 1, \dots, 2p$.

Let $\operatorname{Re} \alpha_k - \mu \neq \frac{1}{2}$ for all k, and let q be the number of points c_k at which $\operatorname{Re} \alpha_k - \mu > \frac{1}{2}$.

Then:

1) if $p > q$, the operator S is right invertible in $H^0_\mu(\Gamma, \rho)$ and $\dim \ker S = p - q$;
2) if $p < q$, the operator S is left invertible in $H^0_\mu(\Gamma, \rho)$ and $\dim \operatorname{coker} S = q - p$;
3) if $p = q$, the operator S is invertible in $H^0_\mu(\Gamma, \rho)$.

The complement $C\Phi_A$ to the Φ-set of the operator S is the union of $2p$ arcs of Z-circumferences, each of which joins the points -1 and 1 and passes through one of the corresponding points $i \operatorname{ctg}[\pi(\operatorname{Re} \alpha_k - \mu)]$ or $-i \operatorname{ctg}[\pi(\operatorname{Re} \alpha_k - \mu)]$, depending on whether c_k is the end or the beginning of the corresponding arc.

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Received
4 VII 1969

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* $\operatorname{ind} K = \dim \ker K - \dim \operatorname{coker} K$.

Note: Figure translations are in progress. See original paper for figures.

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