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Abstract

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ON THE COHOMOLOGIES OF THE LIE ALGEBRA OF SMOOTH VECTOR FIELDS

In our recent paper (^{1}) we began the study of the cohomologies of the topological Lie algebra of smooth* vector fields on a connected compact orientable smooth manifold. The present note contains several new facts about these cohomologies. The results stated in paragraphs 1–5 are essentially contained in the papers (^{1,2}); the results of paragraphs 6–11 are new. The central result is a complete description of the cohomologies of the Lie algebra on tori of arbitrary dimension, and also on two-dimensional manifolds (paras. 5, 8, 10, 11).

1. Let us recall that the **cohomologies of a topological Lie algebra**** $\mathfrak{g}$ **with coefficients in $\mathbf{R}$** are defined as the cohomologies of the standard complex $\{C^q, d^q\}$, where C^q is the space of continuous skew-symmetric q-linear real-valued functionals on $\mathfrak{g}$, and the homomorphism $d^q : C^q \to C^{q+1}$ is defined by the formula

$$(d^{qP})(\xi_1, \dots, \xi_{q+1}) = \sum_{1 \le s < t \le q+1} (-1)^{s+t-1} P([\xi_s, \xi_t], \xi_1, \dots, \widehat{\xi_s}, \dots, \widehat{\xi_t}, \dots, \xi_{q+1})$$

(here $\xi_1, \dots, \xi_{q+1} \in \mathfrak{g}$, $P \in C^q$). The elements of the space C^q are called q-dimensional chains of the algebra $\mathfrak{g}$. The complex $\{C^q, d^q\}$ is endowed with a canonical multiplicative structure, turning the space

$$H^*(\mathfrak{g}; \mathbf{R}) = \sum_{q \ge 0} H^q(\mathfrak{g}; \mathbf{R})$$

of its cohomologies into a graded (associative) algebra.

We shall denote the standard complex of the topological Lie algebra W_n of formal vector fields at the origin of the coordinate space $\mathbf{R}^n$ (see (^{2}), para. 0.1) by $\{C^q(n), d^q(n)\}$, and the cohomologies of this algebra (with coefficients in $\mathbf{R}$) by

$$\mathfrak{H}^*(n) = \sum_{q \geq 0} \mathfrak{H}^q(n).$$

The topological Lie algebra of smooth vector fields on a smooth manifold M will be denoted by $\mathfrak{A}(M)$; the standard complex corresponding to this algebra will be denoted by $\{C^q(M), d^q(M)\}$, or, more briefly, by $\mathfrak{C}(M)$; and the cohomologies of the algebra $\mathfrak{A}(M)$ (with coefficients in $\mathbf{R}$) by

$$\mathfrak{H}^*(M) = \sum \mathfrak{H}^q(M).$$

2. Let X_n denote the full inverse image of the $2n$-dimensional skeleton of the base of the universal $U(n)$-bundle $(EU(n), p, BU(n))$ under the map p (we have in mind the usual cellular decomposition of the space $BU(n)$ –see, for example, (^{3}), p. 89). Obviously, X_n is a $2n$-connected $n(n+2)$-dimensional cellular complex; in particular, X_1 is the three-dimensional sphere.

For every q there is an equality $\mathfrak{H}^q(n) = H^q(X_n; \mathbf{R})$. Multiplication in the ring $\mathfrak{H}^*(n)$ is trivial, i.e., the product of any two elements of positive dimension is equal to zero.

This theorem constitutes the main content of our paper (^{2}).

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* Smoothness is everywhere understood as belonging to the class C^∞.

** In saying "algebra," "space," etc., we everywhere take the field $\mathbf{R}$ of real numbers as the ground field.

3. The group Diff^n of diffeomorphisms of the space $\mathbb{R}^n$ preserving the origin acts in the obvious way on the algebra W_n, and hence also in the spaces $C^*(n)$, $\mathfrak{H}^*(n)$ of its cochains and cohomologies.

The action of the group Diff^n in $\mathfrak{H}^*(n)$ is trivial. For any element of the space $\mathfrak{H}^{n(n+2)}(n)$ there exists in $C^{n(n+2)}(n)$ a representing cocycle invariant with respect to the group Diff^n. For $q < n(n+2)$, an element of $\mathfrak{H}^q(n)$, generally speaking, cannot be represented by a Diff^n-invariant cocycle, but can always be represented by a cocycle invariant with respect to the group $GL(n, \mathbb{R}) \subset \mathrm{Diff}^n$, composed of linear transformations.

This proposition is easily derived from (^{2}), although it is not explicitly formulated there.

4. Let M be a connected compact orientable smooth n-dimensional manifold. In the complex $\mathcal{C}(M)$ we define a filtration, assigning to a cochain $P \in C^q(M)$ filtration $\leq k$ if, for any smooth vector fields $\xi_1, \dots, \xi_q \in \mathfrak{A}(M)$ with $P(\xi_1, \dots, \xi_q) \neq 0$, there exist points $x_1, \dots, x_k \in M$ such that $\xi_s(x_{j_s}) \neq 0$

for some j_s for each s. The subspace of the space $C^q(M)$ consisting of elements of filtration $\leq k$ is denoted by $C^q_k(M)$. Obviously,

$$0 = C^q_0(M) \subset \cdots \subset C^q_q(M) = \cdots = C^q(M).$$

This filtration is compatible with the differential $d^q(M)$, so that for each k a subcomplex

$$\mathcal{C}_k(M) = \{C^q_k(M),\, d^q(M)\}$$

of the complex $\mathcal{C}(M)$ is defined. The complex $\mathcal{C}_0(M)$ is trivial; the complex $\mathcal{C}_1(M)$ is called diagonal.

The filtration is also compatible with multiplication in the complex $\mathcal{C}(M)$. For details see (^{1}), §§ 1.2–1.5.

5. In § 2 of the paper (^{1}), in the complex $\mathcal{C}_k(M)$ (for each k) a new filtration is introduced. With the aid of this filtration a spectral sequence

$$\{\, {}^{(k)}E^{u,v}_r,\ {}^{(k)}d^{u,v}_r : {}^{(k)}E^{u,v}_r \to {}^{(k)}E^{u+r,v-r+1}_r \,\}$$

is defined, converging to the cohomologies of the quotient complex $\mathcal{C}_k(M)/\mathcal{C}_{k-1}(M)$. The second term of this spectral sequence admits the following description. Denote by M^k the product of k copies of the manifold M, and by M^k_* the part of this product consisting of points $(x_1, \dots, x_k) \in M^k$ among whose coordinates $x_1, \dots, x_k$ there are coincident ones. Put

$$H(k,p,q) = H_p\left(M^k, M^k_*; \bigoplus_{\substack{m_1+\cdots+m_k=q\\ m_1>0,\dots,m_k>0}} (\mathfrak{H}^{m_1}(n) \otimes \cdots \otimes \mathfrak{H}^{m_k}(n))\right) =$$

$$= H^{kn-p}(M^k \setminus M^k_*; \mathbb{R}) \otimes H^q\,(X_n \# \cdots \# X_n, *; \mathbb{R})$$

where in the last expression there are k factors X_n under the brace.

(# denotes the tensor product of spaces, i.e. the direct product with the coordinate cross collapsed to a point; $*$ is the marked point.) In the space $H(k,p,q)$ acts the group $S(k)$ of permutations of k elements (the group acts simultaneously in M^k and in $X_n \# \cdots \# X_n$). The space ${}^{(k)}E^{u,v}_2$ is isomorphic to the space of $S(k)$-invariant elements of the space $H(k,-u,v)$; in particular, ${}^{(k)}E^{u,v}_2$ can be nontrivial only for

$$-kn \leq u \leq 0, \qquad 2n < v \leq n(n+2).$$

For $k = 1$ the spectral sequence

$$\{\, {}^{(k)}E^{u,v}_r,\ {}^{(k)}d^{u,v}_r \,\}$$

converges to the cohomologies of the diagonal complex and is denoted simply by

$$\{\, E^{u,v}_r,\ d^{u,v}_r \,\}.$$

From the proposition formulated above it follows that, for $v > 0$, the equality

$$E_2^{u,v} = H_{-u}(M, \mathbb{R}) \otimes \mathfrak{H}^v(n) = H^{u+n}(M; \mathbb{R}) \otimes H^v(X_n; \mathbb{R})$$

holds.

Proofs of the assertions set forth in this item are contained in §§ 3–8 of the paper (^{1}) and in § 6 of the paper (^{2}).

6. Multiplication

$$C_k^p(M) \otimes C_l^q(M) \to C_{k+l}^{p+q}(M)$$

induces, for each r, multiplication

$${}^{(k)}E_r^{u_1,v_1} \otimes {}^{(l)}E_r^{u_2,v_2} \to {}^{(k+l)}E_r^{u_1+u_2,v_1+v_2},$$

connected with the differential by the formula [{}$^{\{(k+l)\}d\_r}${u__1+u__2,v__1+v__2}(__1 __2) = ({}$^{\{(k)\}d\_r}${u__1,v__1} __1) __2

- (-1)^{u__1+v__1} __1({}$^{\{(l)\}d\_r}${u__2,v__2} __2).]

In particular, there arises a multiplication

$$E_r^{u_1,v_1} \otimes \cdots \otimes E_r^{u_k,v_k} \to {}^{(k)}E_r^{u_1+\cdots+u_k,\, v_1+\cdots+v_k},$$

where the product of the elements

$$\Phi_i \otimes \alpha_i \in H^{n-r_i}(M; \mathbf{R}) \otimes \mathfrak{h}^{q_i+r_i}(n) = E_2^{-r_i,\, q_i+r_i}$$

is equal to the element of the space

$${}^{(k)}E_2^{-r_1-\cdots-r_k,\, q_1+\cdots+q_k+r_1+\cdots+r_k},$$

obtained from

$$\pi(\Phi_1 \otimes \cdots \otimes \Phi_k) \otimes \alpha_1 \otimes \cdots \otimes \alpha_k,$$

where

$$\pi : H^*(M^k, \mathbf{R}) \to H^*(M^k \setminus M_*^k, \mathbf{R})$$

is the homomorphism induced by the inclusion $M^k \setminus M_*^k \to M^k$, symmetrization, and the assignment of the corresponding sign. The homomorphism π, as is easy to show, is an epimorphism, and therefore the multiplication

$$E_2^{u_1,v_1} \otimes \cdots \otimes E_2^{u_k,v_k} \to {}^{(k)}E_2^{u_1+\cdots+u_k,\, v_1+\cdots+v_k}$$

is epimorphic.

7. If the spectral sequence $\{E_r^{u,v}, d_r^{u,v}\}$ is trivial, i.e. $d_r^{u,v} = 0$ for $r \geq 2$, then:

(1) All the spectral sequences $\{{}^{(k)}E_r^{u,v}, {}^{(k)}d_r^{u,v}\}$ are trivial, and hence the cohomology of the complex $\mathfrak{C}_k(M)/\mathfrak{C}_{k-1}(M)$ coincides with ${}^{(k)}E_2$.

(2) Every cohomology class of the complex $\mathfrak{C}_k(M)/\mathfrak{C}_{k-1}(M)$ is represented by a product of k cochains from $\mathfrak{C}_1(M)$.

(3) The product of k cochains from $\mathfrak{C}_1(M)$ is cohomologous to zero in $\mathfrak{C}(M)$ if and only if it is cohomologous to zero in $\mathfrak{C}_k(M)/\mathfrak{C}_{k-1}(M)$.

(4) The space $\mathfrak{H}^*(M)$ is the direct sum of the cohomology spaces of the complexes $\mathfrak{C}_k(M)/\mathfrak{C}_{k-1}(M)$, $k = 1, 2, \dots$.

(5) The ring $\mathfrak{H}^*(M)$ is generated (multiplicatively) by the cohomology classes of the complex $\mathfrak{C}_1(M)$; in particular, it has a finite number of generators.

This proposition is easily derived from what was said in item 6 (in the proof of assertion (3) one uses the triviality of the multiplication in the ring $\mathfrak{H}^*(n)$—see item 2).

8. In the case when the spectral sequence of the diagonal complex is trivial, Proposition 7 gives a complete description of the ring $\mathfrak{H}^*(M)$: in this case there is an isomorphism

$$\mathfrak{H}^*(M) = \sum_{k,u,v} {}^{(k)}E_2^{u,v},$$

and the multiplication

$$\mathfrak{H}^*(M) \otimes \mathfrak{H}^*(M) \to \mathfrak{H}^*(M)$$

is generated by the multiplications

$${}^{(k)}E_2 \otimes {}^{(l)}E_2 \to {}^{(k+l)}E_2.$$

Thus, the computation of the ring $\mathfrak{H}^*(M)$ is reduced, in the indicated case, to the computation of the cohomology rings of the spaces $M^k \setminus M^k_*$ and X_n, which is already carried out by standard topological methods.

9. The element

$$\Phi \otimes a \in H^{n-r}(M; \mathbf{R}) \otimes \mathfrak{h}^{q+r}(n) = E_2^{-r,\, q+r}$$

can be represented by a cochain of the complex $\mathfrak{C}_1(M)$ in the following way. Fix on the manifold M a finite covering by charts $\{U_\nu; x_1^{(\nu)}, \dots, x_n^{(\nu)}\}$ (here $x_1^{(\nu)}, \dots, x_n^{(\nu)}$ are local coordinates in U_ν) and a partition of unity γ_ν subordinate to this covering. Also fix a closed differential form φ in the cohomology class Φ, and a cocycle $a \in C^{q+r}(n)$ representing the cohomology class α. Let $\xi_1, \dots, \xi_q \in \mathfrak{A}(M)$. For each point $x \in U_\nu$ the vector fields $\xi_1, \dots, \xi_q$ determine, in view of the existence of a coordinate system in U_ν, elements $\xi_1(x), \dots, \xi_q(x) \in W_n$. Put

$$\psi_\nu = \sum_{1 \le i_1 < \dots < i_r \le n} a(\xi_1(x), \dots, \xi_q(x), e_{i_1}, \dots, e_{i_r})\, dx_{i_1}^{(\nu)} \wedge \dots \wedge dx_{i_r}^{(\nu)},$$

where $e_1, \dots, e_n$ are the basis vector fields in $\mathbf{R}^n$, and define a differential form ψ of degree r on M as $\sum_\nu \gamma_\nu \psi_\nu$. Assigning to the elements $\xi_1, \dots, \xi_q \in \mathfrak{A}(M)$ the number

$$\int_M \psi \wedge \varphi,$$

we obtain a continuous skew-symmetric q-linear real functional λ on $\mathfrak{A}(M)$, i.e. an element of $C^q(M)$. It is verified directly that: (a) the cochain λ belongs to the diagonal complex; b) the element it represents from E_0 belongs to the kernel of the differentials d_0 and d_1 and determines in E_2 the element

$\Phi \otimes a$ (independently of the choice of local coordinates, partition of unity, the form φ, and the cochain a); (c) to the cochain $[d^q(M)]\lambda$ there corresponds the number, assigned to vector fields $\xi_1, \dots, \xi_{q+1} \in \mathfrak{A}(M)$,

$$\int_M \widetilde{\psi} \wedge \varphi, \qquad \text{where} \quad \widetilde{\psi} = \sum_\nu \gamma_\nu \, d\psi'_\nu,$$

$$\psi'_\nu = \sum_{1 \le i_1 < \dots < i_{r-1} \le n} a(\xi_1(x), \dots, \xi_{q+1}(x), e_{i_1}, \dots, e_{i_{r-1}}) \, dx^{(\nu)}_{i_1} \wedge \dots \wedge dx^{(\nu)}_{i_{r-1}}.$$

10. It is clear that if the cochain λ constructed in item 8 is a cocycle of the complex $\mathcal{C}(M)$, then the element $\Phi \otimes a \in E_2^{-r,q+r}$ belongs to the kernel of all differentials. Here are two cases when λ is a cocycle for trivial reasons: (a) the element $a \in C^{q+r}(n)$ (see item 8) is invariant with respect to the group Diff_n; (b) the element a is invariant with respect to the group $GD(n, \mathbf{R})$, and the transition functions from one set of local coordinates $x_1^\nu, \dots, x_n^\nu$ to another are all linear. In both cases the forms ψ'_ν (as well as the forms ψ_ν) agree on overlaps of charts, i.e. they serve as restrictions of a global form ψ', and therefore

$$\widetilde{\psi} = \left(\sum_\nu \gamma_\nu \right) d\psi' = d\psi', \qquad \{[d^q(M)]\lambda\}(\xi_1, \dots, \xi_{q+1}) = \int_M d\psi' \wedge \varphi = 0.$$

Let us give two consequences of this observation.

(1) *If M is a torus (the product of n circles), then the spectral sequence $\{E_r^{u,v}, d_r^{u,v}\}$ is trivial.*

This follows from what was said above (case (b)) and item 3.

(2) *If $n = 2$, then the spectral sequence $\{E_r^{u,v}, d_r^{u,v}\}$ is trivial.*

Indeed, for $n = 2$ the space $E_2^{u,v}$ can be nontrivial only for $u = -2, -1, 0$ and $v = 5, 7, 8$. By dimensional considerations, only the differential

$$d^{-2,8} : \; E_2^{-2,8} \to E^{0,7}$$

can be nontrivial, and it is trivial by what was said above (case (a)) and item 3.

11. The final solution of the problem of computing the ring $\mathfrak{H}^*(M)$ for the case when M is a torus of arbitrary dimension or a two-dimensional manifold is obtained by a simple comparison of the results of items 5, 8, and 10. For example, the ring $\mathfrak{H}^*(S^2)$ is generated by the generators $a_1, a_2, a_3, a_4, a_5;\ b_1, b_2, b_3, b_4, b_5$ of dimensions $3, 3, 5, 6, 6;\ 5, 5, 7, 8, 8$, respectively, subject, besides the usual anticommutativity relations, to the relations

$$b_i b_j = 0 \quad (i, j \text{ arbitrary}); \qquad a_i b_j + a_j b_i = 0 \quad (i \neq j); \qquad a_4 b_4 = a_5 b_5 = 0.$$

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