THE HILBERT SPACE OF S. L. SOBOLEV AS A VARIATIONAL STRUCTURE

MATHEMATICS

1970

SovietRxiv

Abstract

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UDC 513.88:513.83

MATHEMATICS

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THE HILBERT SPACE OF S. L. SOBOLEV AS A VARIATIONAL STRUCTURE

(Presented by Academician S. L. Sobolev on 2 III 1970)

The theory of normed semiordered lineals ([2–4]) does not encompass certain spaces of differentiable functions, since the requirement of the existence of an exact bound sometimes turns out to be impossible to fulfill or is not compatible with the norm. In the present paper the concept is introduced of such a partially ordered Hilbert space in which every finite set has, among its upper bounds, a least one in the variational sense, compatible with the norm. This space is called a variational structure. It is a generalization of the concept of a KH-space ([5]).

In one class of Hilbert vector structures a method is indicated for introducing another order (in a stronger sense), allowing the presence of variational bounds.

It is proved that all the constructions are applicable to the space of S. L. Sobolev $W = W_2'$ ([1]). The order has a number of natural properties: the introduced comparability of functions entails their pointwise comparability and is compatible with convergence in norm (theorem); in W there are variational bounds of finite families of functions; if a family has a least upper bound, then the variational upper bound coincides with it; for lower bounds a duality relation is satisfied.

1. Let in a Hilbert space X there be distinguished a closed cone $K \neq \{0\}$, containing zero and having properties 1), 2). Elements $x \in K$ are called positive, $x \geqslant 0$. By definition, $x > 0$ if $x \in K,\ x \neq 0$; $x > y,\ y < x$, if $x - y > 0$. It is required: 1) if elements $x,\ y > 0$, the number $\lambda > 0$, then $x \neq 0,\ x + y > 0,\ \lambda x > 0,\ (x, y) \geqslant 0$; 2) for every element $x \in X$ there exists an element $y \geqslant x,\ 0$. Then the inequalities $0 < x < y$ entail $\|x\| \leqslant \|y\|$; every element $x \in X$ decomposes into a difference of elements of the cone K. In the partially ordered lineal X the usual connections between comparison relations and vector operations hold.

For a finite set F, consisting of n elements x_i (the number n arbitrary), denote by V the set of its upper bounds $v \geqslant x_i$. Denote $V \geqslant F$. The set V is closed by virtue of the closedness of the cone K. It is nonempty: by condition 2) there exist elements $z_i \geqslant x_i,\ 0$; consequently, the element

$$\sum_{k=1}^{n} z_k \in V.$$

Define the functionals

$$G(v) = (v, v), \qquad H(v) = \sum_{i=1}^{n} G(v - x_i)$$

on elements $v \in V$. The functionals are obviously continuous, nonnegative, and nondecreasing on V as a consequence of the monotonicity of the norm for positive elements.

Theorem 1. For any finite set $F \subset X$ there exists a unique upper bound v_0 such that for all $v \in V \geq F$

$$\inf H(v) = H(v_0).$$

In the proof, a minimizing sequence is constructed and the identity

$$H_2(v + w) + nG(v - w) = 2H(v) + 2H(w),$$

is used, where the elements $v, w \in V$, and the functional

$$H_2(u) = \sum_{i=1}^{n} G(u - 2x_i)$$

is defined on the algebraic sum $V + V$.

Definition. The element v_0 that gives the minimum of the functional H on the set V of upper bounds of a finite set F will be called the **variational upper bound** of the set F. The space X will be called a **variational structure**. We retain here the notation: $v_0 = \sup x_i$, in particular, $v_0 = x \vee y$. The variational lower bound is $\inf x_i = -\sup(-x_i)$. In particular, $x \wedge y$.

If F has a least upper bound (in the usual sense), then the element v_0 necessarily coincides with it by virtue of the nondecreasing nature of the functional H on the set V and the uniqueness of the variational bound. Therefore the variational structure is (in the class of Hilbert spaces) an intermediate object between partially ordered and semiordered lineals.

Let us note some properties of bounds. Stretching, contraction: $\sup \lambda x_i = \lambda v_0$ for $\lambda > 0$; $\sup \lambda x_i = \lambda \inf x_i$ for $\lambda < 0$. Shift: $\sup(x_i + x) = v_0 + x$ for all $x \in X$. Further,

$$x \vee y + x \wedge y = x + y.$$

The usual notation is introduced:

$$x_+ = x \vee 0, \quad x_- = (-x) \vee 0, \quad |x| = x_+ + x_-, \quad x_i \vee 0 = x_n^+.$$

It is established that

$$x_+ \leq |x|, \quad -|x| \leq x \leq |x|, \quad x = x_+ - x_-, \quad x_+ \wedge x_- = 0.$$

The representation of an element x in the form $y - z$, where $y \wedge z = 0$, is unique. Further, $|x| = x \vee -x = x_+ \vee x_-$, $\||x|\| = \|x\|$.

2. Let now X be an H-space and a K-lineal with cone M of positive elements. We require:

a) disjointness of elements entails their orthogonality;

b) the set $N \neq \{0\}$, where $N = \{y : (x, y) \geq 0 \ \forall x \in M\}$;

c) the cone M is closed.

For condition c) to hold, it is sufficient that

d) the set $\{x : (x, y) \geq 0 \ \forall y \in N\} = M$.

From condition a) the relations follow

$$\|x_+\|, \ \|x_-\| \leq \| \ |x| \ \| = \|x\|. \tag{1}$$

Let us also note that $N \subset M$. Indeed, if $y \in N$, then $y_- \in M$, $(y, y_-) \geq 0$, $(y_+, y_-) \geq (y_-, y_-)$. But $y_+ \perp y_-$ by condition a). Consequently, $y_- = 0$, $y \in M$.

We introduce in X another order, turning X into a variational structure. Choose in $N \neq \{0\}$ some subset P, any pair p, q of elements of which (including $p = q$) gives the scalar product $(p, q) \geq 1$ $(p, q \in P \subset N)$. Obviously, the elements $\lambda p \in P$ for $\lambda \geq 1$.

Form the sets

$$Q = \{y + \mu p : \ y \in M, \ p \in P, \ \mu \geq \|y\|\},$$

$$R = \left\{ \sum_{i=1}^{n} x_i : \ x_i \in Q \right\},$$

where the numbers n are arbitrary. It is clear that $\lambda Q \subset Q,\ \lambda R \subset R$ for $\lambda \geq 0$; $R + R = R$. Setting $y = 0,\ \mu = 1$, we find $P \subset Q \subset R$. Moreover, $R \subset M$,

$$(v, w) \geq 0 \qquad (v, w \in R). \tag{2}$$

We shall call the closure of the set R a cone K, $\overline{R} = K$.

Theorem 2. X is a variational structure with cone K. Moreover, $K \subset M$, i.e., comparability acquires a stronger meaning.

The inequality $(x, y) \geq 0$ for $x, y \in K$ is obtained from inequalities of the form (2) by passage to the limit. The inclusion $K \subset M$ follows from the relations $R \subset M = \overline{M}$. Since in the K-lineal X with cone M every element x is representable as the difference of elements of the cone M, the proof of the existence of an element $y \in K$ such that $y - x \in K$ reduces to the case $x \in M$. But such an element is, for example, $y = x + \mu p$, where the number $\mu \geq \|x\|, 1$.

Theorem 3. The order is compatible with convergence: if $|x_n| \leq |y_n|,\ y_n \to 0$, then $x_n \to 0$.

It suffices to establish that the convergence $x_n \to 0$ implies $x_n^+ \to 0$ (here the moduli and positive parts of elements are taken in the variational sense, see item 1). Take a fixed element $p \in P$, construct the faces $y_n = x_n \vee 0,\ z_n = (-x_n) \vee 0$ (now in the sense of the K-lineal X with cone M) and the elements $v_n = y_n + \|x_n\| p$. By virtue of relations (1) we have $\|y_n\| \leq \|x_n\|$. Therefore $v_n \to 0$. Since $v_n \geq 0,\ v_n - x_n = z_n + \|x_n\| p \geq 0$ with respect to the cone K, it follows, according to the definition of the variational face x_n^+, that

$$H(x_n^+) \leq H(v_n), \qquad G(x_n^+) \leq G(x_n^+ - x_n) + G(x_n^+) \leq G(v_n - x_n) + G(v_n) \to 0,$$

and hence $x_n^+ \to 0$.

3. We pass to the space $W = W_2'$.

Theorem 4. If as M one takes the family of all nonnegative functions from W, then W satisfies all the requirements of item 2 (including conditions a)–d)) and, consequently, is a variational structure with cone K. Comparability with respect to the cone K entails pointwise comparability and is compatible with convergence in norm.

Thus, at first comparability is given a pointwise meaning (with respect to the cone M) and then a stronger one (with respect to the cone K). The composition of the new cone K depends on the choice of the family P, which may be taken depending on the conditions of the particular problem being solved in the space W.

In the proof of Theorem 4 it is first established that the pointwise-defined face $x \vee 0$ for $x \in W$ is also contained in W, and hence W is a K-lineal with cone M. Pointwise disjointness of the functions x, y implies pointwise disjointness of their generalized derivatives and, consequently, orthogonality of the elements x, y in the H-space W. The domain of definition of the functions is denoted by D, its boundary by S. The set N is nontrivial; in particular, it contains all functions with nonpositive Laplacian satisfying the boundary condition $x|_S = 0$.

To prove fulfillment of condition d) of item 2, suppose there exists an element $x \notin M$ such that $(x, y) \geq 0$ for all $y \in N$. Then the function $x < 0$ on some subset $E \subset D$ of positive measure. Find in W_2^2 the solution of the problem $y|_S = 0$ for the equation $\Delta y = -\chi$, where χ is the characteristic function of the set E. Then, by Green' s formula,

$$(x, y) = -\int_D x \Delta y \, dD = \int_E x \, dD < 0,$$

which contradicts the inequality $(x, y) \geq 0$. Such is the scheme of the proof of Theorem 4.

Let us note that if the inequality $x > 0$ is understood as $x \neq 0$ and the nonnegativity of all first derivatives of the functions x, then this comparability will be compatible with the norm in W, but for some boundary-value problems it loses its meaning:

Thus, for example, in the class of functions satisfying the boundary condition $x|_S = 0$, there is no element $x > 0$ in the latter sense.

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Received
24 II 1970

CITED LITERATURE

[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Novosibirsk, 1962. [2] N. Bourbaki, *Algebra. Polynomials and Fields. Ordered Groups*, "Nauka," 1965. [3] L. Fuchs, *Partially Ordered Algebraic Systems*, Moscow, 1965. [4] B. Z. Vulikh, *Introduction to the Theory of Semiordered Spaces*, Moscow, 1961. [5] S. N. Slugin, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 29, no. 1, 215 (1965).

Note: Figure translations are in progress. See original paper for figures.