DISTRIBUTION OF PRODUCTS OF SHIFTED PRIME NUMBERS IN ARITHMETIC PROGRESSIONS

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Abstract

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DISTRIBUTION OF PRODUCTS OF SHIFTED PRIME NUMBERS IN ARITHMETIC PROGRESSIONS

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After I. M. Vinogradov created the method of estimates of trigonometric sums with prime numbers (see ([1])), papers appeared ([4], [5]) in which necessary and sufficient conditions were obtained, in terms of estimates of trigonometric sums with prime numbers, for the validity of the quasi-Riemann hypothesis and, consequently, for the corresponding laws of distribution of prime numbers. In the present paper questions are studied that are connected with the distribution of numbers of the form $p(p'+a)$, where p, p' are prime numbers, in arithmetic progressions with increasing difference D. We make essential use of I. M. Vinogradov' s method.

In order to present the essence of the matter most clearly, we consider here only the simplest case of the problem posed; the upper bound for D can be considerably increased (but not beyond n^{χ_1}, where $\chi_1 = 1/(2.5+\omega)$), which, however, is connected with a complication of the proof of the theorem.

In exactly the same way one studies the question of the distribution in arithmetic progressions of numbers of the form $(p^n + a)f(p')$, where p and p' are prime numbers, and f is a polynomial with integer coefficients. Moreover, by the same method one can solve problems on the distribution of prime numbers in arithmetic progressions "on average" and other problems.

Notation. $\omega \in (0, 1/4]$; n is a sufficiently large positive number; D is a prime number, $D \le n^{\chi_0}$, where $\chi_0 = 1/(4.6+\omega)$; $(a, D) = (l, D) = 1$; χ is a Dirichlet character $\bmod D$; $\alpha \in [(1/2+\omega)\ln D/\ln n,\ 1 - 4.1\ln D/\ln n]$; $n_1 \ge n^{1-\alpha}$, $n_2 \ge n^{\alpha}$; p, p' are prime numbers;

$$\pi(x) = \sum_{p \le x} 1; \qquad \pi_2 = \pi_2(n_1, n_2, a, l) = \sum_{\substack{p(p'+a) \equiv l \pmod D \\ p \le n_1,\ p' \le n_2}} 1;$$

$\varepsilon > 0$ is arbitrarily small, not always one and the same; $\psi_1(u)$ and $\psi_2(v)$ are certain functions of u and v, with $|\psi_1(u)| \le u^{\varepsilon}$, $|\psi_2(v)| \le v^{\varepsilon}$.

Theorem. There exists an absolute constant $\gamma > 0$ such that

$$\pi_2 = \frac{1}{\varphi(D)}\pi(n_1)\pi(n_2) + O((n_1 n_2)^{1+\varepsilon} D^{-1-\gamma\omega^2}),$$

where the constant in the O-symbol depends only on ω.

Lemma 1. Let

$$P = \prod_{p \le H} p; \qquad Q = \prod_{H < p \le N} p;$$

s_0 be the largest integer satisfying $H^{s_0} \le N$; $\theta(x)$ be an arbitrary function of x such that $|\theta(x)| \le 1$;

$$S = \sum_{p \le N} \theta(p),$$

$$W_s = \sum_{d_1/P} \cdots \sum_{\substack{d_s/P \\ d_1 \cdots d_s m_1 \cdots m_s \le N}} \sum_{m_1 > 0} \cdots \sum_{m_s > 0} \mu(d_1) \cdots \mu(d_s) \theta(d_1 \cdots d_s m_1 \cdots m_s),$$

$$W'_s = \sum_{y_1/Q} \cdots \sum_{y_s/Q} \theta(y_1 \dots y_s).$$

$$y_1 \dots y_s \leqslant N, \ \mu(y_1 \dots y_s) \ne 0$$

Then, for certain $\alpha_1, \dots, \alpha_{s_0}, \alpha'_1, \dots, \alpha'_{s_0}, c$, depending only on s_0, we have

$$S = \alpha_1 W_1 + \dots + \alpha_{s_0} W_{s_0} + \alpha'_1 W'_1 + \dots + \alpha'_{s_0} W'_{s_0} + cH.$$

The proof of this lemma is the same as that of Lemma 10 in the paper ([2]).

Lemma 2. Let $N \geqslant D^{1/2+\omega}$, $(k, D) = 1$; let χ be a nonprincipal character mod D. Then there exists an absolute constant $\gamma > 0$ such that

$$S_N = \sum_{p \leqslant N} \chi(p + k) \ll N D^{-\gamma\omega^2},$$

where the constant in the sign $\ll$ depends only on ω.

For the proof of this lemma, see ([3]).

Lemma 3. Let $D \leqslant U < U_1 \leqslant 2U, \ D \leqslant V < V_1 \leqslant 2V,$

$$W = \frac{1}{\varphi(D)} \sum_{\chi \bmod D} \left| \sum_{\substack{U<u\leqslant U_1,\ V<v\leqslant V_1 \\ uv\leqslant N}} \psi_1(u)\psi_2(v)\chi(uv) \right|.$$

Then

$$W \ll (UV)^{1+\varepsilon} D^{-1} \ll N^{1+\varepsilon} D^{-1}.$$

Proof of the theorem. We have the equality

$$\pi_2 = \frac{1}{\varphi(D)} \sum_{\chi \bmod D} \sum_{p\leqslant n_1,\ p'\leqslant n_2} \chi(p(p'^{\,j}+a))\overline{\chi}(l) =$$

$$= \frac{1}{\varphi(D)}\pi(n_1)\pi(n_2) + R + O(n_1 n_2 D^{-2}),$$

where

$$|R| \leqslant \frac{1}{\varphi(D)} \sum_{\chi\neq\chi_0} \left| \sum_{p\leqslant n_1} \chi(p) \right| \left| \sum_{p\leqslant n_2} \chi(p+a) \right|.$$

Using Lemma 2, we obtain

$$|R| \leqslant n_2 D^{-\gamma\omega^2} T, \quad \text{where} \quad T = \frac{1}{\varphi(D)} \sum_{\chi\neq\chi_0} \left| \sum_{p\leqslant n_1} \chi(p) \right|. \qquad (*)$$

Putting $N = n_1$, in Lemma 1 take $H = \max(N^{0,1}, \sqrt{D})$ and apply it to the inner sum T. We obtain the inequality

$$T \ll \sum_{1\leqslant s\leqslant s_0} \frac{1}{\varphi(D)} \sum_{\chi\neq\chi_0} (|W_s| + |W_s'|) + H,$$

where $s_0 \leqslant 10$, and the constant in the sign $\ll$ is absolute.

From the definition of the sums W_s' it follows that

$$\frac{1}{\varphi(D)} \sum_{\chi\neq\chi_0} |W_s'| \ll N^{\varepsilon} \sum_{H<d\leqslant N} \sqrt{\frac{N}{d^2}\left(\frac{N}{Dd^2}+1\right)} \ll N^{1+\varepsilon} D^{-1}.$$

It remains to estimate $(1 \leqslant s \leqslant s_0)$

$$T_1 = \frac{1}{\varphi(D)} \sum_{\chi \neq \chi_0} |W_s|.$$

Take $c = 1/60$; apply Lemma 5 of the book ([1]), p. 313, in the formulation given in ([2]), p. 492. All divisors $d \mid P$, $d \leqslant N$, are distributed among $\leqslant D = (\ln N)^{\ln\ln N/\ln(1+c)}$ sets; in addition, the intervals $0 < m_i \leqslant N$, $1 \leqslant i \leqslant s$, are divided into $\ll \ln N$ intervals of the form $M_i < m_i \leqslant$

$\ll M_i' \leqslant 2M_i$. We obtain $\ll D(\ln N)^s$ sums T_2 of the form

$$T_2 = \frac{1}{\varphi(D)} \sum_{\chi \neq \chi_0} \left| \sum_{d_1} \cdots \sum_{m_s} \chi(d_1 \dots d_s m_1 \dots m_s) \right|,$$

where the summation is over the domain $M_i < m_i \leqslant M_i'$, $\varphi_i < d_i \leqslant \varphi_i^{1+c}$, $i = 1, \dots, s$, $m_1 \dots m_s d_1 \dots d_s \leqslant N$.

It is enough to consider the case $M_1 \dots M_s (\varphi_1 \dots \varphi_s)^{1+c} \geqslant N D^{-1/2}$, since otherwise we trivially have $T_2 \ll N^{1+\varepsilon} D^{-1}$.

Denote $M_1 \dots M_s \varphi_1 \dots \varphi_s = \Phi$; then either a) $\Phi = \Phi_1 \Phi_2$, $\Phi_1 \geqslant D$, $\Phi_2 \geqslant D$, $\Phi_1 = M_{i_1} \dots M_{i_r} \varphi_{j_1} \dots \varphi_{j_k}$, or b) the representation a) is impossible.

a) Putting $u = m_{i_1} \dots m_{i_r} d_{j_1} \dots d_{j_k}$, $v = d_1 \dots d_s m_1 \dots m_s u^{-1}$, we obtain

$$T_2 = \frac{1}{\varphi(D)} \sum_{\chi \neq \chi_0} \left| \sum_{\substack{U < u \leqslant U^* \\ uv \leqslant N}} \sum_{V < v \leqslant V^*} \psi_1(u) \psi_2(v) \chi(uv) \right|.$$

Splitting the intervals of variation of u and v into $\ll \ln N$ intervals of the form $U_1 < u \leqslant U_1' \leqslant 2U_1$, $V_1 < v \leqslant V_1' \leqslant 2V_1$, and observing that $U_1 \geqslant D$, $V_1 \geqslant D$, we obtain $\ll \ln^2 N$ sums T_2', to each of which the estimate of Lemma 3 is applicable. Thus, $T_2 \ll N^{1+\varepsilon} D^{-1}$.

b) In this case either 1) $\max_{1 \leqslant i \leqslant s} M_i \geqslant \Phi D^{-1}$, or 2) $\max_{1 \leqslant i \leqslant s} \varphi_i \geqslant \Phi D^{-1}$.

1) Let $\max_{1 \leqslant i \leqslant s} M_i = M_1$, $u = m_1$, $v = m_2 \dots m_s d_1 \dots d_s$. Then $\Phi D^{-1} \leqslant M_1 < u \leqslant M_1'$, $uv = m_1 \dots m_s d_1 \dots d_s \leqslant \Phi(\varphi_1 \dots \varphi_s)^c$, $\varphi_1 \dots \varphi_s \leqslant D$; $v \leqslant D^{1+c}$; therefore T_2 does not exceed $\ll \ln N$ sums T_2' of the form

$$T_2' = \frac{1}{\varphi(D)} \sum_{\chi \neq \chi_0} \left| \sum_{V_1 < v \leqslant V_1'} \psi_2(v) \chi(v) \sum_{M_1 < u \leqslant \min(M_1', N v^{-1})} \chi(u) \right|.$$

Consequently,

$$T_2' \ll D^{1+c}\sqrt{D}\ln D \ll N^{1+\varepsilon}D^{-1}, \qquad T_2 \ll N^{1+\varepsilon}D^{-1}.$$

2) Let $\max_{1\leqslant i\leqslant s}\varphi_i = \varphi_j = \varphi$; then

$$\varphi \geqslant \Phi D^{-1} \geqslant N^{1/(1+c)}D^{-1-1/2(1+c)}.$$

Put $U = D^{3/2}$; then $U < \varphi' \leqslant UH$, $\varphi'\varphi'' = \varphi$, and

$$T_2 = \frac{1}{\varphi(D)}\sum_{\chi\neq\chi_0}\left|\sum_{\varphi'<d'\leqslant\varphi'^{1+c}}\sum_{\varphi''<d''\leqslant\varphi''^{1+c}}\cdots\sum_{M_s<m_s\leqslant M_s'}\chi(d'd''\ldots m_s)\right|$$

$$(d',d'')=1,\ d'd''\ldots m_s\leqslant N$$

$$= \frac{1}{\varphi(D)}\sum_{\chi\neq\chi_0}\left|\sum_{d\leqslant N}\mu(d)\chi(d^2)\sum_{\varphi'd^{-1}<d'\leqslant\varphi'^{1+c}d^{-1}}\chi(d')\times\right.$$

$$\left.\times\sum_{\substack{\varphi''d^{-1}<d''\leqslant\varphi''^{1+c}d^{-1}\\ d'd''\ldots m_s\leqslant Nd^{-2}}}\cdots\sum_{M_s<m_s\leqslant M_s}\chi(d''\ldots m_s)\right|$$

$$\leqslant \frac{1}{\varphi(D)}\sum_{\chi\neq\chi_0}\left|\sum_{d\leqslant\sqrt{D}}K(d)\right| + \frac{1}{\varphi(D)}\sum_{\chi\neq\chi_0}\left|\sum_{d>\sqrt{D}}K(d)\right|, \qquad (**)$$

where

$$K(d) = \mu(d)\chi(d^2)\sum_{\varphi'd^{-1}<d'<\varphi'^{1+c}d^{-1}}\sum_{\varphi''d^{-1}<d''\leq\varphi''^{1+c}d^{-1}}\sum_{M_s<m_s\leq M_s'}\chi(d')\chi(d''\cdots m_s),$$

$$d'd''\cdots m_s \leq Nd^{-2}.$$

For the second sum in the last inequality we have the estimate

$$\ll \sum_{N\geq d>\sqrt{D}}\frac{1}{\varphi(D)}\sum_{\chi\neq\chi_0}|K(d)| = \sum_{N\geq d>\sqrt{D}}\frac{1}{\varphi(D)}\sum_{\chi\neq\chi_0}\left|\sum_{u\leq Nd^{-2}}^{1}\psi_1(u)\chi(u)\right| \ll$$

$$\ll N^{\varepsilon}\sum_{N\geq d>\sqrt{D}}\sqrt{\frac{N}{d^2}\left(\frac{N}{Dd^2}+1\right)} \ll N^{1+\varepsilon}D^{-1}.$$

Let us now consider the first sum on the right-hand side of $(**)$. The summands in $K(d)$ have the form $\chi(u)\chi(v)$, where $u = d'$, $v = d''\cdots m_s$; splitting the intervals of variation of the quantities u and v into intervals, as we did above, we obtain $\ll \ln^2 N$ sums of the form

$$\frac{1}{\varphi(D)} \sum_{\chi \neq \chi_0} \left| \sum_{\substack{U<u\leq U_1 \\ uv \leq Nd^{-2}}} \sum_{V<v\leq V_1} \psi_1(u)\psi_2(v)\chi(uv) \right|, \qquad (***)$$

where $U \gg \varphi' D^{-1/2} \gg D$; $V \gg \varphi'' D^{-1/2} \cdots M_s = D^{-1/2}\Phi\varphi'^{-1-c} \gg D$.

Applying Lemma 3 to $(***)$, we obtain

$$|K(d)| \ll N^{1+\varepsilon} D^{-1} d^{-2}; \qquad \sum_{d<\sqrt{D}} |K(d)| \ll N^{1+\varepsilon} D^{-1}.$$

Consequently, $T_2 \ll N^{1+\varepsilon} D^{-1}$. Thus, for T we have obtained the estimate

$$T \ll N^{1+\varepsilon} D^{-1}.$$

From this estimate and $(*)$ the assertion of the theorem follows.

Remark 1. If one repeats the proof of the theorem, making explicit the meaning of the estimates with ε, then for some absolute constant $c > 0$ one obtains

$$\pi_2 = \frac{1}{\varphi(D)} \pi(n_1)\pi(n_2) + O\left(n_1 n_2 e^{c(\ln\ln n_1 n_2)^2} D^{-1-\gamma\omega^2}\right).$$

Remark 2. One can obtain an asymptotic formula for π_2 for any $D \geq 1$.

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Note: Figure translations are in progress. See original paper for figures.