THEOREMS ON THREE CYLINDERS FOR SOLUTIONS OF LINEAR EVOLUTION QUASI-ELLIPTIC EQUATIONS

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Abstract

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THEOREMS ON THREE CYLINDERS FOR SOLUTIONS OF LINEAR EVOLUTION QUASI-ELLIPTIC EQUATIONS

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Here, with the aid of special algebraic conditions, we single out classes of evolution equations of arbitrary order containing, for example, besides parabolic equations, equations of the type of equations of transverse vibrations of elastic plates and rods, for which L_1-estimates of solutions are obtained that do not contain the values of the solutions on the initial hyperplane, and, in the case of cylinders adjacent to the boundary, the values of the solutions on the boundary hypersurface. The investigation covers equations having power singularities on the initial hypersurface (boundary hypersurface).

The methods used by us are very close to those set forth in ([1]) and are their natural continuation and development.

1. Conditions. Notation. Consider the equation

$$\mathcal{L}u \equiv \mathcal{L}\left(t, x; \frac{\partial}{\partial t}, D_x\right) u \equiv \sum_{k_0 p+|k|\le m} a'_{k_0 k}(t,x) \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k u(t,x) = f(t,x). \qquad (1)$$

We introduce a number of conditions necessary for the formulation of the results:

A. The operator $\mathcal{L}$ has the Lagrange adjoint operator

$$\mathcal{L}^*\left(t, x; \frac{\partial}{\partial t}, D_x\right) \equiv \sum_{k_0 p+|k|=m} a_{k_0 k}(t,x) \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k + \sum_{k_0 p+|k|<m} a_{k_0 k}(t,x) \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k \equiv$$

$$\equiv P_0(t, x; \partial/\partial t, D_x) + P_1(t, x; \partial/\partial t, D_x); \qquad p \ge 1; \quad \nu = m/p.$$

B. The polynomials

$$\mathrm{B}_1.\quad P_0(t,x;0,\sigma),$$

$$\mathrm{B}_2.\quad P_0(t,x;\sigma_{n+1}^p,\sigma).$$

$$\mathrm{B}_{3,\beta}.\quad Q_0(t,x,\sigma')=\sum_{\mu=1}^{\nu}\prod_{l=1}^{\mu-1}\left(1-\frac{l}{\beta}\right)\frac{1}{\mu!}\,\frac{\partial^\mu}{\partial\lambda^\mu}P_0(t,x;\lambda,\sigma)\Big|_{\lambda=0}\sigma_{n+1}^{p\mu},$$

$$p\geq\nu,\qquad \sigma'=(\sigma_{n+1},\sigma_1,\dots,\sigma_n).$$

$$\mathrm{B}_4^{(\alpha)}.\quad T_0^{(\alpha)}(t,x,\sigma')\equiv\sum_{\mu=0}^{\nu}\prod_{l=1}^{m-p\mu-1}\left(1-\frac{l}{\alpha}\right)\frac{1}{\mu!}\,\frac{\partial^\mu P_0}{\partial\lambda^\mu}\Big|_{\lambda=0}\sigma_{n+1}^{p\mu},\qquad \alpha\geq m.$$

$$\mathrm{B}_5^{(\gamma)}.\quad Q_0^{(\gamma)}(t,x;\sigma')\equiv\sum_{q=0}^{m}\prod_{l=1}^{q-1}\left(1-\frac{l}{\gamma}\right)\frac{1}{q!}\,\frac{\partial^q P_0}{\partial\sigma_n^q}\Big|_{\sigma_n=0}\sigma_n^q,\qquad \gamma\geq m.$$

$$\mathrm{B}_{6,\beta}^{(\gamma)}.\quad R_{0\beta}^{(\gamma)}(t,x;\sigma')\doteq\sum_{\mu=0}^{\nu}\sum_{q=0}^{m}\prod_{l=1}^{\mu-1}\left(1-\frac{l}{\beta}\right)\prod_{s=1}^{q-1}\left(1-\frac{s}{\gamma}\right)\frac{1}{\mu!q!}\times$$

$$\times\frac{\partial^{\mu+q}P_0}{\partial\lambda^\mu\partial\sigma_n^q}\Big|_{\sigma_n=0,\lambda=0}\sigma_{n+1}^{p\mu}\sigma_n^q$$

are uniformly elliptic with constant uniform ellipticity $\delta_0,\delta_1,\delta_\beta,\delta^{(\alpha)},\delta_1^{(\gamma)},\delta_\beta^{(\gamma)}$, respectively.

C. The ranges of values of the polynomials

$\mathrm{C}_1.\ P_0(t,x;0,\sigma)$. $\quad$ $\mathrm{C}_2.\ P_0(t,x;\sigma_{n+1}^p,\sigma)$. $\quad$ $\mathrm{C}_{3,\beta}.\ Q_{03}(t,x;\sigma')$.

$\mathrm{C}_4^{(\alpha)}.\ T_0^{(\alpha)}(t,x;\sigma')$. $\quad$ $\mathrm{C}_5^{(\gamma)}.\ Q_0^{(\gamma)}(s,x;\sigma')$. $\quad$ $\mathrm{C}_{6,\beta}^{(\gamma)}.\ R_{0\beta}^{(\gamma)}(t,x;\sigma')$

for any real σ' lies in the cone (sector) $K_{\varphi_1}:\ |\arg z|\leq\varphi_1<\pi/2$ of the complex z-plane.

E. The coefficients $a_{k_0k}(t,x)$ of the polynomial $\mathcal{L}^*(t,x;\lambda,\sigma)$:
E_1. Are uniformly bounded by a constant e_1.
E_2. For $k_0p+|k|=m$ they are uniformly continuous.

E_3. $a_{k_0 k}(bt, b^{1/p}x) b^{(m-k_0 p-|k|)/p}$, $b \in (0,1)$, are uniformly bounded by a constant e_3.

Condition B_1 means ellipticity of the spatial part of the operator $\mathcal{L}$; condition B_2 will be called the quasi-ellipticity of this operator. Let us note that, for sufficiently large $\beta; \alpha; \beta$ and γ, conditions $B_{3,\beta}$, $C_{3,\beta}$, $B_4^{(\alpha)}$, $C_4^{(\alpha)}$, $B_{5,\beta}^{(\gamma)}$, $C_{5,\beta}^{(\gamma)}$ follow from conditions B_2, C_2, respectively. If the coefficients of the polynomial $P_0(t, x; \lambda, \sigma)$ are real, then conditions C (with $\varphi_1 = 0$) follow from conditions B. It is useful to note that for $\beta = \nu$, $\gamma = m$:

$$Q_{0\nu} = \sum_{\mu=0}^{\nu} C_\nu^\mu \frac{\partial^\mu}{\partial \lambda^\mu} P_0 \bigg|_{\lambda=0} \sigma_{n+1}^{p\mu}, \qquad Q_0^{(m)} = \sum_{q=0}^{m} C_m^q \frac{\partial^q}{\partial \sigma_n^q} P_0 \bigg|_{\sigma_n=0} \sigma_n^q;$$

$$R_{0\nu}^{(m)} = \sum_{\mu=0}^{\nu} \sum_{q=0}^{m} C_\nu^\mu C_m^q \frac{\partial^{\mu+q}}{\partial \lambda^\mu \partial \sigma_n^q} P_0 \bigg|_{\lambda=0, \sigma_n=0} \sigma_{n+1}^{p\mu} \sigma_n^q.$$

In the case of an equation of first order in t:

$$P_0(t, x; \lambda, \sigma) = \lambda + \sum_{|k|=m} a_k(t, x) \sigma^k, \qquad \nu = 1, \quad m = p,$$

$$Q_{01}(t, x; \sigma') \equiv P_0(t, x; \sigma_{n+1}^p, \sigma) = \sigma_{n+1}^m + \sum_{|k|=m} a_k(t, x) \sigma^k,$$

$$R_{01}^{(m)}(t, x; \sigma') = Q_0^{(m)}(t, x; \sigma') = \sigma_{n+1}^m + \sum_{q=0}^{m} C_m^q \frac{\partial^q}{\partial \sigma_n^q} P_0(t, x; 0, \sigma) \bigg|_{\sigma_n=0} \sigma_n^q.$$

In particular, for $m = 2$,

$$Q_{01} = \sigma_{n+1}^2 + \sum_{i,j=1}^{n} a_{ij}(t, x) \sigma_i \sigma_j,$$

$a_{ij}(t, x)$ are real functions, and conditions $B_5^{(2)}$, $B_{6,1}^{(2)}$ mean the ellipticity of the polynomials $Q_0^{(2)}$, $R_{01}^{(2)}$; the best condition invariant under rotations in the space $(x_1, x_2, \dots, x_n)$ for this is the inequality $\lambda_1 < (n + 2 + 2\sqrt{2})\lambda_n$, where λ_1 is the largest and λ_n the smallest characteristic number of the matrix $(a_{ij})_1^n$.

If the polynomial $P_0(t, x, \sigma_{n+1}^p, \sigma)$ with real positive coefficients contains only even powers of $\sigma_{n+1}, \sigma_1, \sigma_2, \dots, \sigma_n$, then all the conditions described above are automatically satisfied. In particular, they are satisfied for the polynomial

$$\sigma_{n+1}^2 + a^2(\sigma_1^2 + \dots + \sigma_n^2)^2,$$

corresponding to the equation of transverse elastic vibrations. If the operator has the form

$$\mathcal{L}u = \sum_{k_0 p+|k|\le m} (-1)^{k_0+|k|} \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k (a_{k_0 k}(t,x)u),$$

then it is sufficient to require only boundedness of the coefficients $a_{k_0 k}(t,x)$.

Define the cone (sector) K of the complex z-plane by the inequality

$$K = K_{\varphi_2} = \{z;\ |\arg z| \le \varphi_2 < \pi/2 - \varphi_1\}.$$

With the aid of the cone K we represent the function $u(t,x)$ in the form $u(t,x) = u^+(t,x) - u^-(t,x)$, where $u^+(t,x) = u(t,x)$ if $u \in K$; $u^+(t,x) = 0$ if $u \notin K$. Denote

$$\Pi_{(\tau,T),R} = \{(t,x),\ \tau < t < T,\ |x| < R\};$$

$$\Pi^{(\eta)}_{(\tau,T),R} = \Pi_{(\tau,T),R} \cap \{x_n > \eta\}, \qquad \Pi^{(0)}_{(0,T),R} = \Pi^+_{T,R};$$

$$\Pi_{(0,T),R} = \Pi_{T,R}; \qquad M(r) = r \text{ for } r \in [0,1); \qquad M(r) = 1 \text{ for } r \ge 1,$$

$$\|u\|_{(\tau,T),R;\beta} = \iint_{\Pi_{(\tau,T),R}} M[t^\beta]|u|\,dt\,dx;$$

$$\{u\}_{(\tau,T),R,\eta;\beta,\gamma;q} = \iint_{\Pi^{(\eta)}_{(\tau,T),R}} M[(t^\nu + |x|^m)^q]\, M[t^\beta]\, M[x_n^\gamma]|u|\,dt\,dx;$$

$$\|u\|_{(0,T),R;\beta} = \|u\|_{T,R;\beta}; \qquad \{u\}_{(0,T),R,0;\beta,\gamma,q} = \{u\}_{T;R;\beta;\gamma;q};$$

$$\langle u\rangle_{(-T_1,T),R;\alpha} = \iint_{\Pi_{(-T_1,T),R}} (R^2 - x^2)^\alpha |u|\,dt\,dx.$$

2. Theorems on three cylinders

Theorem 1. The conditions A_1, B_1, C_1 are satisfied in $\overline{\Pi}_{T,2}$, $B_{3,\beta}, C_{3,\beta}$ in $\overline{\Pi}_{\varepsilon_0,2}$; E_3 in $\overline{\Pi}_{Tb^{-1},\,2b^{-1/p}}$ (if one assumes E_2 in $\overline{\Pi}_{T,2}$, then C_1 is superfluous, and $B_{3,\beta}, C_{3,\beta}$ need only be assumed for $t = 0$).

Let $u(t,x)$ be a weak solution of equation (1) in $\Pi_{T,2}$. Then there exist positive constants h_1, b_1, λ_2, depending only on $n, m, p, \delta_0, \delta_\beta, \varphi_1, e_3, \varepsilon_0$, such that for every $h \in (0, h_1)$ the estimate holds

$$\|u\|_{T-h,\,1+b_1 h^{1/p};\,\beta-\nu} \le \lambda_1(\|u\|_{(\varepsilon_0,T),1;\,\beta-\nu} + \|u^-\|_{T,2;\,\beta-\nu} + \|f\|_{T,2;\,\beta}).$$

Theorem 2. The conditions A, E_1 are satisfied in $\Pi_{(-T_1,T),R}$; $B_4^{(\alpha)}, C_4^{(\alpha)}$ in $\Pi_{(-T,0],R}$ (if one assumes E_2 in $\Pi_{(-T_2,0],R}$, then $B_4^{(\alpha)}, C_4^{(\alpha)}$ need only be assumed for $t=0$).

Let $u(t,x)$ be a weak solution of equation (1) in $\Pi_{(-T_1,T),R}$; then there exist positive constants t_0, λ_2, depending only on $n, m, p, \delta^{(\alpha)}, \varphi_1, e_1, \varepsilon_0, R$, such that

$$\langle u\rangle_{(-t_0,T),R;\alpha-m} \le \lambda_2(\langle u\rangle_{T,R;\alpha-m} + \langle u^-\rangle_{(-T_1,T),R;\alpha-m} + \langle f\rangle_{(-T_1,T),R,\alpha}).$$

Theorem 3. The conditions A, B_1, C_1 are satisfied in $\overline{\Pi}^+_{T,2}$; $B_{3,\beta}, C_{3,\beta}$ in $\overline{\Pi}^+_{\varepsilon_0,2}$, $B_5^{(\gamma)}, C_5^{(\gamma)}$ in $\overline{\Pi}^+_{T,2} \cap \{x_n \in [0,\varepsilon_0]\}$; $B_{6,\beta}^{(\gamma)}, C_{6,\beta}^{(\gamma)}$ in $\overline{\Pi}^+_{\varepsilon_0,2} \cap \{x_n \in [0,\varepsilon_0]\}$; E_3 in $\overline{\Pi}^+_{Tb^{-1},\,2b^{-1/p}}$ (if E_2 is assumed in $\overline{\Pi}^+_{T,2}$, then C_1 is superfluous, $B_{3,\beta}, C_{3,\beta}$ need only be assumed for $t=0$; $B_5^{(\gamma)}, C_5^{(\gamma)}$ for $x_n=0$; $B_{6,\beta}^{(\gamma)}, C_{6,\beta}^{(\gamma)}$ for $t=0,\ x_n=0$).

Let $u(t,x)$ be a weak solution of equation (1) in $\Pi^+_{T,2}$; then there exist positive constants h_2, b_2, λ_3, depending only on $n, m, p, \delta_0, \delta_1^{(\gamma)}, \delta_\beta^{(\gamma)}, \delta_\beta, \varphi_1, \varepsilon_0, e_3$, such that for every $h \in (0,h_2)$ the estimate holds

$$\{u\}_{T-h,\,1+b_2h^{1/p};\,\beta-\nu,\,\gamma-m;\,1} \le \lambda_3(\{u\}_{(\varepsilon_0,T),1,\varepsilon_0,0;0;0} + \{u^-\}_{T,2;\,\beta-\nu,\,\gamma-m;\,1} + \{f\}_{T,2;\,\beta,\nu;\,0}).$$

The proofs of Theorems 1 and 3 are carried out by covering the initial cylinder with lunettes, semi-lunettes, and quarters of lunettes, in which the norm of the positive part is estimated, under the assumptions made, on the basis of the "sign-definiteness" of $\mathcal{L}^*\Phi$; as Φ one takes, respectively, functions of the form

$$t^\beta(1-x^2-t^2)^\alpha, \qquad t^\beta x_n^\gamma(1-x^2-t^2)^\alpha, \qquad x_n^\gamma(1-x^2-t^2)^\alpha,$$

while Theorem 2 is proved by covering the whole cylinder $\Pi_{T,R}$ from below by a cylinder of small height and using the function

$$t^\beta(R^2-x^2)^\alpha.$$

We note that for $\beta=\nu,\ \alpha=m,\ \gamma=m$, Theorems 1 and 2 give L_1-estimates of solutions; for $\beta>\nu,\ \alpha>m,\ \gamma>m$, they give estimates in a space with the corresponding weight. In this case the solutions are allowed to have singularities on the initial and boundary hyperplanes of arbitrary power order (for arbitrary β, α, γ). If β, α, γ are large, then all the estimates are valid under the assumption of quasellipticity of the polynomial $P_0(t,x;\lambda,\sigma)$ and the fulfillment of condition C_2.

Theorems 1, 2, and 3 have various applications. We shall present some of them.

3. Membership of positive solutions of quasi-elliptic equations in the space $L_1^{(\beta,\gamma)}(Q_T)$

Let $Q_T=(0,T)\times\Omega$, where Ω is a bounded domain with boundary S belonging to the class C^{m-l+n} (2). Denote by $\nu(x)$ the inward normal to S at the point x.

Then, as usual, the polynomial $P_0(t,x;\sigma^p_{n+1},\sigma)$ can be written in coordinates corresponding to the half-space $\nu\cdot x>0$ in the form

$$P_0(t,x;\sigma^p_{n+1},\theta'+\nu\theta_n)\equiv P_{0\nu}(t,x;\sigma^p_{n+1},\theta),\qquad \theta'=(\theta_1,\dots,\theta_{n-1},0),$$

and from the polynomial $P_{0\nu}$ define $Q^{(\gamma)}_{0\nu},R^{(\gamma)}_{0\nu,\beta}$; the conditions B and C for these polynomials are assumed to hold uniformly in ν. Let Ω_{ε_0} denote the set of points of the domain Ω lying at distance greater than ε_0 from S, $Q_{(\varepsilon_0,T),\varepsilon_0}=$ $=(\varepsilon_0,T)\times\Omega_{\varepsilon_0}$, $\rho(x)$ is the distance from the point x to S. Introduce the space $L_1^{(\beta,\gamma,q)}(Q_T)$ of functions for which the norm

$$\{u\}_{Q_T;\beta,\gamma}=\iint_{Q_T}M\big[(t^\nu+\rho(x)^m)^q\big]\,M[t^\beta]\,M[\rho(x)^\gamma]\,|u(t,x)|\,dt\,dx$$

is finite.

With the aid of Theorems 1 and 3 one establishes

Theorem 4. Suppose that conditions A, B_1, C_1, E_1 are satisfied in $\overline{Q}_T$; conditions $B_{3,\beta}$, $C_{3,\beta}$ in $\overline{Q}_{\varepsilon_0}$; $B_5^{(\gamma)}$, $C_5^{(\gamma)}$ in $\overline{Q}_T/Q_{(\varepsilon_0,T),\varepsilon}$; $B_5^{(\gamma)}$, $C_5^{(\gamma)}$ in $\overline{Q}_{\varepsilon_0}\cap\{x_n\in[0,\varepsilon_0)\}$. If:
1) $u(t,x)$ is a weak solution of equation (1) in Q_T; 2) $f(t,x)\in L_1^{(\beta,\gamma;0)}(Q_T)$;
3) $u^-(t,x)\in L_1^{(\beta-\nu,\gamma-m;1)}(Q_T)$, then $u(t,x)\in L_1^{(\beta-\nu,\gamma-m;1)}(Q_{T_1})$ for any $T_1<T$, and the estimate

$$\{u\}_{Q_{T_1};\beta-\nu,\gamma-m;1}\le\lambda_4(\{u\}_{Q_{(\varepsilon_0,T);\varepsilon_0,0,0;0}}+\{u^-\}_{Q_T;\beta-\nu,\gamma-m;1}+\{f\}_{Q_T;\beta,\gamma;0})$$

is valid; λ_4 depends only on λ (from Theorem 3), ε_0, and $T-T_1$.

With the aid of Theorems 1 and 3 one establishes propositions on the growth of solutions defined in an unbounded cylinder (where the coefficients may grow with the growth of the spatial coordinates), generalizing and strengthening the results stated in (1). This makes it possible, in particular, on the basis of the usual uniqueness theorems, to establish theorems on the coincidence of solutions with identical initial conditions under estimates of their negative components.

4. On solutions in an infinite (in t) tube

On the basis of Theorem 2 one establishes

Theorem 5. Suppose that conditions A, E_1, $B_4^{(\alpha)}$, $C_4^{(\alpha)}$ are satisfied in $\Pi_{(-\infty,0),R}$. If $u(t,x)$ is a weak solution of the equation $\mathcal{L}u=0$ in $\Pi_{(-\infty,0),R}$, $\langle u\rangle_{(-1,0),R;\alpha-m}<M_1$, $\langle u^-\rangle_{(t,0),R;\alpha-m}\le M_2e^{c|t|}$, $t<0$, then

$$\langle u\rangle_{(t,0),R;\alpha-m} \le K_1(c,\lambda_2)(M_1+M_2)e^{(C+\ln 2\lambda_2/t_0)|t|},$$

where λ_2, t_0 are from Theorem 2.

If the coefficients $P_0(t,x;\sigma_{n+1}^p,\sigma)$ are real, then for sufficiently large α all the conditions of Theorem 5 reduce to its ellipticity. If $P_0(0,\sigma)$ is an elliptic polynomial, then the ellipticity of $P_0(\sigma_{n+1}^p,\sigma)$ is not only sufficient but also necessary in order that a positive solution of equation (1) with the principal group of terms could grow no faster than $e^{c|t|}$ as $t\to-\infty$.

5. Degenerate equations

All the theorems remain valid for degenerate equations. The auxiliary functions and the method of proof are the same. The algebraic conditions are derived in an analogous way. For illustration, let us write down the construction of the equation in the case of degeneration on a hyperplane. The operator $\mathcal{L}^*$ is assumed to be of the form:

$$\mathcal{L}^* = \sum_{j=0}^{m} t^{-j/p} \sum_{k_0p+|k|=m-j} a_{k_0k}(t,x)\frac{\partial^{k_0}}{\partial t^{k_0}}D_x^k + P_1 \equiv \sum_{j=0}^{m} t^{-j/p}P_{0j} + P_1,$$

the $a_{k_0k}(t,x)$ are bounded; the coefficients of the polynomial P_1 have the property that $b_{k_0k}(t,x)t^{(m-|k_0k|)/p-k_0-\varepsilon_1}$, $\varepsilon_1>0$, are bounded. The principal polynomial is

$$Q_{0\beta} = \sum_{\mu=0}^{m}\beta^{-j/p}\prod_{s=1}^{\mu-1}\left(1-\frac{s}{\beta}\right)\frac{1}{\mu!}\,\frac{\partial^\mu}{\partial\lambda^\mu}P_{0j}\Big|_{\lambda=0}\sigma_{n+1}^{p\mu+j}.$$

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