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Abstract

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MATHEMATICS

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ON SPACES CLOSE TO NORMAL AND THEIR BICOMPACT EXTENSIONS

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1. Statement of the question. Among the bicompact extensions of a space X, three deserve special attention: the space βX (Čech–Stone), the space ωX (Wallman), and the space $\omega_\varkappa X$ (Ponomarev-Zaitsev). The spaces βX and ωX are bicompact extensions of the space X in all cases when they are defined, i.e. ωX for all T_1-spaces, and βX for all Tikhonov (completely regular) spaces. The space $\omega_\varkappa X$, defined by V. I. Ponomarev and studied in detail by V. I. Zaitsev ([1–3]), is remarkable in that (as Zaitsev proved*) it is always the limit space of a finite inverse spectrum of the space X. Zaitsev also proved that $\omega_\varkappa X$ is a bicompact extension for a broad class of T_λ-spaces X introduced by him, including, in particular, all regular spaces**.

It has long been known that the spaces βX and ωX coincide if and only if the space X is normal. Zaitsev proved that the spaces $\omega_\varkappa X$ and βX coincide for a certain class of quasinormal spaces defined by him, and only for them. In the present note the question is solved: for which spaces X do the spaces ωX and $\omega_\varkappa X$ coincide? It turns out that for this it is necessary and sufficient that the space X be seminormal in the sense that we shall now establish.

2. Seminormal spaces. V. I. Zaitsev calls quasinormal those regular spaces in which any two disjoint nonintersecting π-sets are contained in disjoint open sets (which, obviously, may always be assumed to be $\varkappa o$-sets).

Definition 1. Every T_1-space in which any two disjoint closed sets are contained in disjoint π-sets is called **seminormal*****.

Remark 1. Obviously, we obtain the same class of spaces if we require only that for any two disjoint closed sets A_1 and A_2 there exist a π-set containing A_1 and not intersecting A_2.

An immediate consequence of this definition is

Proposition 1. *A space X is normal if it is simultaneously quasinormal and seminormal.*

V. I. Zaitsev observed that there exist quasinormal spaces which are not normal —such are, in particular, all nonnormal

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* On all questions concerning the space $\omega_{\varkappa}X$, see ([1−3]).

** T_{λ}-spaces may be defined as those T_1-spaces in which every point and every closed set not containing it lie in disjoint π-sets. Here π-sets are defined by V. I. Zaitsev as sets that are intersections of a finite number of canonical closed sets. (A set is called canonically closed, or a $\varkappa a$-set, if it is the closure of its open kernel. Sets complementary to $\varkappa a$-sets are called canonically open, or $\varkappa o$-sets.)

*** Thus, every seminormal space is T_{λ} and, consequently, all the more semiregular; the transition from T_{λ}-spaces to seminormal ones is entirely analogous to the transition from regular spaces to normal ones. Every bicompact semiregular space is seminormal.

extremely disconnected spaces, and also, for example, the well-known Niemytzki space. By a slight modification of Niemytzki' s construction one can also obtain an example of a Tychonoff space that is not quasinormal. As Zaitsev showed, every quasinormal space is completely regular; consequently, the class of quasinormal spaces, being contained in the class of Tychonoff spaces and including all normal spaces, does not coincide with either of these classes. On the other hand, it follows from Proposition 1 that a nonnormal quasinormal space cannot be seminormal; at the same time a Tychonoff seminormal (but nonnormal) space (and such spaces exist) is never quasinormal.

Proposition 2. *Not every seminormal space is Hausdorff; not every Hausdorff seminormal space is regular.*

3. Main theorem. *In order that the spaces ωX and $\omega_x X$ coincide (up to a homeomorphism leaving fixed all points $x \in X$), it is necessary and sufficient that the space X be seminormal; in this case $\omega_x X$ is also seminormal.*

Lemma. *Let X be a space for which $\omega_x X$ is a (bicompact) extension, and let $i : X \to \omega_x X$ be the natural mapping of the space X into the space $\omega_x X$. Let the sets $A \subseteq X$ and $B \subseteq X$ be closed in X. The closures $[iA]$ and $[iB]$ in $\omega_x X$ are disjoint if and only if there exist disjoint π-sets $C \supseteq A$ and $D \supseteq B$ in X.*

Proof of the lemma. The condition is sufficient. Indeed, suppose there exist disjoint π-sets

$$C = \bigcap_{k=1}^{m} C_k, \qquad D = \bigcap_{k=1}^{n} D_k$$

(where C_k and D_k are χa-sets in X). For any χa-set $Q \subseteq X$, denote by $\widetilde{Q}$ the set of all such $\xi \in \omega_x X$ that $Q \in \xi$. The collection of all $\widetilde{Q}$ forms, by definition, a closed base of the space $\omega_x X$.

The sets

$$\widetilde{C} = \bigcap_{k=1}^{m} \widetilde{C}_k \quad \text{and} \quad \widetilde{D} = \bigcap_{k=1}^{n} \widetilde{D}_k$$

are disjoint, $\widetilde{C} \supseteq [iA]$, $\widetilde{D} \supseteq [iB]$, hence $[iA] \cap [iB] = \Lambda$.

The condition is necessary. Suppose $[iA] \cap [iB] = \Lambda$. We have

$$[iA] = \bigcap_{\widetilde{A}_\alpha \supseteq iA} \widetilde{A}_\alpha,$$

$$[iB] = \bigcap_{\widetilde{B}_\beta \supseteq iB} \widetilde{B}_\beta, \quad \text{and therefore, by assumption,} \quad \bigcap_{\alpha,\beta} \widetilde{A}_\alpha \cap \widetilde{B}_\beta = \Lambda.$$

It follows from this (since $\omega_x X$ is bicompact) that the system $\{A_\alpha\} \cup \{B_\alpha\}$ is not centered. Hence there exist such $\widetilde{A}_{\alpha_1}, \dots, \widetilde{A}_{\alpha_m}$ and $\widetilde{B}_{\beta_1}, \dots, \widetilde{B}_{\beta_n}$ that

$$\bigcap_{k=1}^{m} A_{\alpha_k} \cap \bigcap_{k=1}^{n} B_{\beta_k} = \Lambda.$$

At the same time

$$\bigcap_{k=1}^{m} \widetilde{A}_{\alpha_k} \supseteq iA \quad \text{and} \quad \bigcap_{k=1}^{n} \widetilde{B}_{\beta_k} \supseteq iB.$$

It is easy to see that from $\widetilde{A}_\alpha \supseteq iA$ it follows that $A_\alpha \supseteq A$, so that

$$C = \bigcap_{k=1}^{m} A_{\alpha_k} \supseteq A,$$

and analogously

$$D = \bigcap_{k=1}^{n} B_{\beta_n} \supseteq B.$$

The sets C, D are the required π-sets: their intersection is empty, since otherwise one would have

$$\bigcap_{k=1}^{m} \widetilde{A}_{\alpha_k} \cap \bigcap_{k=1}^{n} B_{\beta_k} \neq \Lambda.$$

The lemma is proved.

We pass to the proof of the main theorem.

1°. Let X be seminormal and let $\zeta \in \omega X$. Denote by $\chi\zeta$ the system of χa-sets contained in ζ. We shall prove that $\chi\zeta \in \omega_x X$. Indeed, otherwise there would be such a χa-set $C \subset X$, $C \notin \xi$, that $\{C\} \cup \chi\zeta$ is a centered system. But from $C \notin \xi$ follows the existence of such an $A \in \xi$ that $A \cap C = \Lambda$. From the seminormality of the space X it followsensures the existence of such χa-sets $D_1, \dots, D_n$, such that

$$D \equiv \bigcap_{k=1}^{n} D_k \supset A, \quad D \cap C = \Lambda.$$

But from $D \supset A \in \xi$ it follows that $D \in \xi$, hence $D_k \in \xi$ for all $k = 1, 2, \dots, n$, and

$$C \cap D_1 \cap \dots \cap D_n = \Lambda,$$

which contradicts the centeredness of the system $C \cup \chi\xi$. Thus we have constructed a mapping

$$\chi : \omega X \to \omega_\chi X,$$

which is evidently a mapping onto all of $\omega_\chi X$.

Let us prove that the mapping $\chi : \omega X \to \omega_\chi X$ is one-to-one. In the contrary case, some system $\xi \in \omega_\chi X$ could be extended to two distinct $\xi_1 \in \omega X$, $\xi_2 \in \omega X$, $\xi_1 \cap \xi_2 \supset \xi$. Since $\xi_1 \neq \xi_2$, there exist disjoint closed subsets $B \in \xi_1$, $B_2 \in \xi_2$, and therefore, by the seminormality of X, there exist χa-sets $D_1, \dots, D_n$ such that

$$B_1 \subset D \equiv \bigcap_{k=1}^{n} D_k \subset X \setminus B_2.$$

The system $\{D_n\} \cup \xi$ is centered and consists of χa-sets. By maximality of the system ξ we have $D_k \in \xi$ for all $k = 1, 2, \dots, n$. But

$$\bigcap_{k=1}^{n} D_k \cap B_i = \Lambda,$$

contrary to the fact that $D_1, \dots, D_n$ are contained in ξ_2. A contradiction.

We prove the continuity of the mapping $\chi : \omega X \to \omega_\chi X$. Recall that a closed base of the space $\omega_\chi X$ is the collection of all $\widetilde{C}$, where C runs through the family of all χa-sets in X, while a closed base of the space ωX is the collection of all Φ_F (where Φ_F is the set of all $\xi \in \omega X$, $\xi \ni F$, and F runs through the family of all closed sets in X). Evidently,

$$\chi^{-1}\widetilde{C} = \Phi_C$$

for every χa-set $C \subset X$. Thus the preimage under χ of any element $\widetilde{C}$ of the closed base of the space $\omega_\chi X$ is a closed set of the space ωX. The continuity of the mapping

$$\chi : \omega X \to \omega_\chi X$$

follows from this.

Analogously, the continuity of the mapping

$$\chi^{-1} : \omega_\chi X \to \omega X$$

will be proved if we verify that, for every closed $F \subset X$, the preimage

$$(\chi^{-1})^{-1}\Phi_F = \chi\Phi_F$$

under the mapping of the set Φ_F is closed in $\omega_\chi X$. But every closed $F \subset X$ is an intersection of some χa-sets C_α:

$$F = \bigcap_\alpha C_\alpha$$

(this follows from the fact that X, being seminormal, is, a fortiori, semiregular). Hence

$$\chi\Phi_F = \chi\Phi_{\bigcap_\alpha C_\alpha} = \bigcap_\alpha \chi\Phi_{C_\alpha} = \bigcap_\alpha \widetilde{C}_\alpha,$$

which proves the required assertion. Thus χ is a topological mapping of the space ωX onto $\omega_\chi X$, evidently leaving fixed all points of X. The first part of the theorem (sufficiency) is proved. We proceed to the second part.

2°. Let $\omega X = \omega_\chi X$ and let X not be seminormal. Then in X there are disjoint closed sets A_1 and A_2 such that every π-set D containing the set A_1 necessarily intersects A_2. Since, by assumption, $\omega_\chi X$ is a bicompact extension of the space X, we may apply the lemma, by virtue of which

$$[iA_1] \cap [iA_2] \neq \Lambda.$$

On the other hand, from $A_1 \cap A_2 = \Lambda$ it follows that also

$$\Phi_{A_1} \cap \Phi_{A_2} = \Lambda,$$

and moreover

$$\Phi_{A_1} = [jA_1], \quad \Phi_{A_2} = [jA_2],$$

where j is the natural mapping

$$j : X \to \omega X,$$

and hence

$$[jA_1] \cap [jA_2] = \Lambda.$$

If there existed a topological mapping j of the space ωX onto $\omega_\chi X$, leaving fixed all points of X, then we would have

$$f([jA_1] \cap [jA_2]) = f([jA_1]) \cap f([jA_2]) = [iA_1] \cap [iA_2] = \Lambda.$$

The contradiction obtained completes the proof of the theorem.

Remark 2. The question arises whether one can simplify the definition of a seminormal space by replacing in it π-sets by χa-sets. The answer to this question is negative: under this replacement we obtain a class of spaces different from the seminormal ones. The analogous question as applied to quasinormal spaces has not yet been resolved.

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Note: Figure translations are in progress. See original paper for figures.