ON THE INDEX OF CERTAIN CLASSES OF INTEGRAL OPERATORS

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Abstract

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MATHEMATICS

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ON THE INDEX OF CERTAIN CLASSES OF INTEGRAL OPERATORS

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Operators of the form

$$H\varphi \equiv \varphi(t) + \sum_{j=1}^{n} \int_{-\infty}^{\infty} a_j(t,\tau) h_j(t-\tau)\varphi(\tau)\,d\tau = f(t), \qquad -\infty < t < \infty, \quad (1)$$

are considered, where $h_j(t) \in \mathcal{L}_1(-\infty,\infty)$, $\varphi(t),\ f(t) \in \mathcal{L}_p(-\infty,\infty)$, $1 \leq p \leq \infty$, and $a_j(t,\tau)$ belong to a certain class of essentially bounded functions measurable in the plane (Definition 3). The main result is formulated in Theorem 1. The results obtained in no. 2 are then applied to the study of a Riemann boundary-value problem with integral terms and of a certain class of integral equations with kernel of the type of a homogeneous function. The present paper is directly adjacent to the preceding work of the authors ([12]), in which the case of degeneration of the functions $a_j(t,\tau)$ was studied:

$$a_j(t,\tau) = \sum_{k=1}^{n} a_{kj}(t) b_{kj}(\tau).$$

A class of equations close to (1) was considered in ([7]). Some special cases of equation (1) were considered in the works ([3–6]). We also note that, in the case of continuity of the functions $a_j(t,\tau)$, our Theorem 1 can also be proved with the aid of the results of I. B. Simonenko on the theory of operators of local type ([8]).

No. 1. The class $M^{\sup}(\widetilde{R}_2)$. We define the class of functions $a_j(t,\tau)$ admissible in equation (1). Roughly speaking, this will be the class of functions having (in a certain sense) at least one of the repeated limits at each of the infinitely remote points $(-\infty,-\infty)$ and $(+\infty,+\infty)$. Let us pass to the precise definition. Denote by $\widetilde{R}_1$ the line R_1 with two adjoined infinitely remote points. Denote by $\widetilde{R}_2$ the plane R_2, completed by the infinitely remote points $(+\infty,+\infty)$, $(-\infty,-\infty)$. As usual, $M = M(R_1)$, $M(R_2)$ will denote the corresponding class of essentially bounded measurable functions.

Definition 1. $\varphi(t) \in M^{\sup}(\widetilde{R}_1)$, if $\varphi(t) \in M(R_1)$ and there exist constants c_+, c_- such that*

$$\lim_{n\to\infty} \sup_{|t|>n} \theta(\pm t)|\varphi(t) - c_\pm| = 0, \qquad \text{where } \theta(t) = \tfrac{1}{2}(1 + \operatorname{sign} t).$$

We shall denote $c_\pm = \varphi(\pm\infty)$.

Definition 2. We shall say that a measurable essentially bounded function $a(t,\tau)$ has the value $a(+\infty,+\infty)$, if there exists a function $b(x) \in M^{\sup}(\widetilde{R}_1)$ such that $b(+\infty) = a(+\infty,+\infty)$ and either

$$\lim_{n\to\infty} \sup_{0<\tau<\infty} \sup_{t>n} |a(t,\tau) - b(\tau)| = 0, \tag{2}$$

* Everywhere in what follows, sup means ess sup.

or

$$\lim_{n\to\infty} \sup_{0<t<\infty} \sup_{\tau>n} |a(t,\tau) - b(t)| = 0. \tag{3}$$

The value $a(-\infty,-\infty)$ is defined analogously. We note that Definition 2 is correct in the sense that if $a(t,\tau)$ has the value $a(+\infty,+\infty)$ simultaneously both in the sense of (2) and in the sense of (3), then it is one and the same.

Definition 3. $a(t,\tau) \in M^{\sup}(\widetilde{R}_2)$ if $a(t,\tau) \in M(R_2)$ and the values $a(+\infty,+\infty)$, $a(-\infty,-\infty)$ exist in the sense of Definition 2. It is clear that the class $M^{\sup}(\widetilde{R}_2)$ contains the class $C(\widetilde{R}_2)$ of continuous functions determined by the properties: 1) $a(t,\tau)$ is bounded on R_2 and continuous at every finite point; 2) one of the iterated limits $\lim_{\tau\to\infty}\lim_{t\to\infty} a(t,\tau)$, $\lim_{t\to\infty}\lim_{\tau\to\infty} a(t,\tau)$ exists, and the inner limit is uniform (with respect to τ, $0<\tau<\infty$, and with respect to t, $0<t<\infty$, respectively); an analogous limit exists as $t,\tau \to -\infty$. We note that Definition 3 imposes no requirements on the function $a(t,\tau)$ in the second and fourth quadrants, apart from membership in $M(R_2)$.

No. 2. **The main theorem.** Let $h_j(t) \in \mathcal{L}_1(-\infty,\infty)$ and $a_j(t,\tau) \in M^{\sup}(\widetilde{R}_2)$.

Theorem 1. In order that the operator H be a Noether operator in $\mathcal{L}_p(-\infty,\infty)$, $1 \le p \le \infty$, it is necessary and sufficient that

$$\sigma(\lambda)^\pm = 1 + \sum_{j=1}^{n} a_j(\pm\infty,\pm\infty)\mathcal{H}_j(\lambda) \neq 0, \quad \text{where} \quad \mathcal{H}(\lambda) = \int_{-\infty}^{\infty} h(t)e^{i\lambda t}\,dt.$$

The index of the operator H is computed by the formula

$$\varkappa_{\mathfrak{B}_p}(H) = -\frac{1}{2\pi}\Delta\left[\arg\frac{\sigma(\lambda)^+}{\sigma(\lambda)^-}\right]_{-\infty}^{\infty}. \qquad (4)$$

The proof of the theorem is based on the following lemma.

Lemma 1. If $h(t) \in \mathcal{L}_1(-\infty,\infty)$ and $a(t,\tau) \in M^{\sup}(\widetilde{R}_2)$, then the operators

$$\int_{-\infty}^{\infty}[a(t,\tau) - a(\infty,\infty)\theta(t) - a(-\infty,-\infty)\theta(-t)]h(t-\tau)\varphi(\tau)\,d\tau,$$

$$\theta(t)\int_{-\infty}^{\infty}a(t,\tau)\theta(-\tau)h(t-\tau)\varphi(\tau)\,d\tau$$

are completely continuous in $\mathcal{L}_p(-\infty,\infty)$, $1 \le p \le \infty$.

No. 3. Consider the following boundary-value problem

$$\Phi^+(t) + \int_{-\infty}^{\infty}a(t,\tau)h_1(t-\tau)\Phi^+(\tau)\,d\tau =$$

$$= G(t)\Phi^-(t) + \int_{-\infty}^{\infty}b(t,\tau)h_2(t-\tau)\Phi^-(\tau)\,d\tau + f(t), \qquad -\infty < t < \infty, \qquad (5)$$

where $\Phi^\pm(z)$ are analytic functions in the half-planes $\operatorname{Im} z > 0$, $\operatorname{Im} z < 0$, respectively, representable by a Cauchy-type integral with density from $\mathcal{L}_p(-\infty,\infty)$; $p > 1$ ($\Phi^\pm(t) \in \mathcal{L}_p^\pm$). It is assumed that 1) $h_1(t)$, $h_2(t) \in \mathcal{L}_1(-\infty,\infty)$; 2) $a(t,\tau)$, $b(t,\tau) \in M^{\sup}(\widetilde{R}_2)$, with $a(+\infty,+\infty) = a(-\infty,-\infty)$ and $b(+\infty,+\infty) = b(-\infty,-\infty)$; 3) $G(t) \in M^{\sup} \cap A_p$ and $G(-\infty) = G(+\infty)$ (for the definition of the class A_p, see ([2])). In particular, one may take $G(t)$ to be a function continuous on the closed axis, $G(t) \neq 0$.

It is known that in the case of a finite contour and Fredholm kernels the index of a problem of the form (5) (as well as of more general integro-differential

problems does not depend on the integral (Fredholm) terms (see (1), p. 362). This, it turns out, is also true for problem (5), which contains integral terms with a difference kernel. However, the condition of normal solvability will depend on the integral terms. We shall also indicate a case when problem (5) is solvable in closed form.

Theorem 2. *If* $1 + a(\infty,\infty)\mathcal{H}_1(x) \neq 0$ *for* $0 \le x \le \infty$ *and* $G(\infty) + b(\infty,\infty)\mathcal{H}_2(x) \neq 0$, $-\infty \le x \le 0$, *where*

$$\mathcal{H}_j(x) = \int_{-\infty}^{\infty}h_j(t)e^{itx}\,dt, \qquad j = 1,2,$$

then problem (5) is Noetherian in $\mathcal{L}_p^{\pm}$, and its index is equal to the index of the coefficient $G(t)$.

By virtue of the lemma, problem (5) differs only by completely continuous terms from the problem

$$A\Phi \equiv \Phi^+(t) + a(\infty,\infty)\int_{-\infty}^{\infty} h_1(t-\tau)\Phi^+(\tau)\,d\tau -$$
$$-G(t)\left[\Phi^-(t) + \frac{b(\infty,\infty)}{G(\infty)}\int_{-\infty}^{\infty} h_2(t-\tau)\Phi^-(\tau)\,d\tau\right] = f(t). \qquad (6)$$

Denoting

$$H_1\Phi^+ \equiv a(\infty,\infty)\int_{-\infty}^{\infty} h_1(t-\tau)\Phi^+(\tau)\,d\tau = \Phi_1^+(t),$$
$$H_2\Phi^- \equiv \frac{b(\infty,\infty)}{G(\infty)}\int_{-\infty}^{\infty} h_2(t-\tau)\Phi^-(\tau)\,d\tau = \Phi_1^-(t),$$

we see that (6) reduces to the successive solution of the Riemann problem

$$\Phi^+(t) + \Phi_1^+(t) = G(t)\left[\Phi^-(t) + \Phi_1^-(t)\right]$$

and of convolution-type integral equations (on the whole axis) in the class of analytic functions. In other words, there is the representation $A = B\cdot C$, where

$$B = \tfrac{1}{2}(I+S) + \tfrac{1}{2}G(I-S), \qquad C = \tfrac{1}{2}(I+H_1)(I+S) + \tfrac{1}{2}(I+H_2)(I-S),$$

G is the operator of multiplication by the function $G(t)$, and

$$S\varphi = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tau)\,d\tau}{\tau - t}.$$

By the assumptions of the theorem, the operator C is invertible.

Naturally, problem (6) is solved in closed form. Consequently, problem (5) is solvable in closed form if $a(t,\tau) = \text{const}$, $b(t,\tau) = \gamma G(t)$, $\gamma = \text{const}$. Note that equation (6) generalizes an equation of the form $\varphi + \lambda S\varphi + H_1\varphi = f$, considered in $\mathcal{L}_2(-\infty,\infty)$ by T. I. Savel' eva (9).

No. 4. Consider the equation

$$K\psi \equiv \psi(x) + \int_0^a \gamma(x,y)k(x,y)\psi(y)\,dy = g(x), \qquad 0 < x < a, \qquad (7)$$

where $k(x,y)$ is a homogeneous function of arbitrary order α: $k(\lambda x,\lambda y) = \lambda^\alpha k(x,y)$, and it is assumed that there exists a number β such that one of the summability conditions is satisfied:

$$\int_0^\infty |k(1,y)|\frac{dy}{y^\beta} < \infty, \qquad \int_0^\infty |k(x,1)|\frac{dx}{x^{1-\beta}} < \infty. \qquad (8)$$

The function $\gamma(x,y)$ will belong to a certain subclass of measurable functions in the fundamental square, bounded everywhere except, possibly, at the origin of the coordinates.

We seek solutions in the weighted space

$$\mathcal{L}_p^\beta = \{\psi : x^{\beta-1/p}\psi(x) \in \mathcal{L}_p(0,a)\}.$$

If in (8) the admissible values are $0 \le \beta \le 1$, then equation (7) may be considered in all $\mathcal{L}_p(0,a)$, $1 \le p \le \infty$.

Denote

$$Q = \{x, y : 0 < x < a,\ 0 < y < a\}.$$

Definition 4. We shall say that $\omega(x,y) \in M^{\sup}(Q)$ if:

1) $\omega(x,y)$ is a measurable function, essentially bounded on Q;
2) $\omega(x,y)$ has the value $\omega(0,0)$, defined analogously to the value $a(\infty,\infty)$ in Definition 2.

Theorem 3. Let $\gamma_1(x,y) = x^{1+\alpha}\gamma(x,y) \in M^{\sup}(Q)$. In order that the operator K be a Noether operator in the space $\mathcal{L}_p^\beta$, $1 \le p \le \infty$, it is necessary and sufficient that

$$\sigma(\lambda) = 1 + \gamma_1(0,0)\mathfrak{M}(i\lambda - \beta + 1) \ne 0,$$

$$-\infty \le \lambda \le \infty, \qquad \mathfrak{M}(s) = \int_0^\infty k(1,y)y^{s-1}\,dy.$$

The index of the operator K is computed by the formula

$$\varkappa_{\mathcal{L}_p^\beta}(K) = -\frac{1}{2\pi}\Delta\left[\arg\sigma(\lambda)\right]_{-\infty}^{\infty}.$$

Theorem 3 is established by reducing equation (7) to equation (1). For simplicity, it is formulated for a kernel $k(x,y)$ satisfying the first of the summability conditions (8), and is easily carried over to the case when $k(x,y)$ satisfies the second of conditions (8). We note that an analogous result for $\alpha = -1$ and $\gamma(x,y) \in C(Q)$ was previously obtained by another method in the works of L. G. Mikhailov ([10,11]). Finally, a theorem analogous to Theorem 3 can be obtained for the case $a = \infty$, and also for an equation more general than (7),

$$K\psi \equiv \psi(x) + \sum_{j=1}^{n}\int_0^a \gamma_j(x,y)k_j(x,y)\psi(y)\,dy = g(x), \qquad 0 < x < a.$$

In conclusion, let us consider the example

$$A\psi \equiv \psi(x) + \int_0^\infty \frac{\gamma(x,y)}{x+y}\psi(y)\,dy = f(x), \qquad x > 0,$$

where $\gamma(x,y) \in M^{\sup}(Q)$, $f(x), \psi(x) \in \mathcal{L}_p(0,\infty)$, $1 < p < \infty$. Let $\gamma_0 = \gamma(0,0)$, $\gamma_\infty = \gamma(\infty,\infty)$–values in the sense of Definition (2). The Noether condition has the form: $\gamma_0, \gamma_\infty > -1/\pi$ for $p = 2$, and $\gamma_0, \gamma_\infty \neq -\frac{1}{\pi}\sin\frac{\pi}{p}$ for $p \neq 2$, while the index $\varkappa$ of the operator A, when this condition is fulfilled, is equal to

$$\varkappa = \begin{cases} 0, & \text{if } \gamma_0, \gamma_\infty > -\sin(\pi/p)/\pi \text{ or } \gamma_0, \gamma_\infty < -\sin(\pi/p)/\pi, \\ \operatorname{sign}(p-2), & \text{if } \gamma_0 > -\sin(\pi/p)/\pi,\ \gamma_\infty < -\sin(\pi/p)/\pi, \\ \operatorname{sign}(2-p), & \text{if } \gamma_0 < -\sin(\pi/p)/\pi,\ \gamma_\infty > -\sin(\pi/p)/\pi. \end{cases}$$

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Note: Figure translations are in progress. See original paper for figures.