EXACT CONDITIONS FOR LOCALIZATION AND CONVERGENCE OF FOURIER SERIES WITH RESPECT TO FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE BELTRAMI OPERATOR

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Abstract

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EXACT CONDITIONS FOR LOCALIZATION AND CONVERGENCE OF FOURIER SERIES WITH RESPECT TO FUNDAMENTAL SYSTEMS OF FUNCTIONS OF THE BELTRAMI OPERATOR

(Presented by Academician A. N. Tikhonov, July 8, 1969)

In the paper ([1]) the notion of a fundamental system of functions (f.s.f.) of the Beltrami operator B in a subdomain Ω (or on a Riemannian submanifold Ω) was introduced, and a number of important properties of fundamental functions were established. In the present paper, relying on the properties mentioned, we establish, in a certain sense, final conditions for localization and uniform convergence of Fourier series with respect to an arbitrary f.s.f. of the Beltrami operator.

The principal result of the present paper is the following assertion.

Main theorem. *Let $N \geq 2$; let G be an arbitrary N-dimensional domain; let $\{u_k(x)\}$ be an arbitrary f.s.f. of the Beltrami operator B in any subdomain Ω of it, with finite points of accumulation allowed for the fundamental numbers $\{\lambda_k\}$. Let the coefficients g^{ik} of the Beltrami operator B belong to the class $C^{(2)}(\bar{\Omega}) \cap C^{(2[N/4]+1)}(\Omega)$. Then, if the function $f(x)$, finite in the domain Ω, belongs in this domain to the class $W_2^{(N-1)/2}(\Omega)$, and in some domain D contained in Ω ** belongs to the class $\overset{\circ}{W}{}^p_\alpha(D)$ for $\alpha = (N-1)/2,\ p > 2N/(N-1)$ for odd N and $\alpha > (N-1)/2,\ p = 2N/(N-1)$ for even N, then, uniformly with respect to x in every strictly interior subdomain Ω' of the domain Ω,*

$$\lim_{\lambda \to +\infty} \sum_{\lambda_k \leq \lambda} f_k u_k(x) = f(x). \tag{1}$$

(Here f_k denotes the Fourier coefficient

$$\int_\Omega f(x) u_k(x) \sqrt{g(x)}\, dx$$

*of the function $f(x)$ with respect to the system $\{u_k(x)\}$**.)*

If the fundamental numbers $\{\lambda_k\}$ have no finite points of accumulation, then instead of (1) one may write

$$\lim_{n\to\infty}\sum_{k=1}^{n} f_k u_k(x) = f(x),$$

assuming here that the fundamental numbers are numbered in increasing order.

Remark. Instead of the domains G and Ω, in defining an f.s.f. one may take a closed N-dimensional Riemannian manifold G and a sufficiently smooth compact N-dimensional Riemannian submanifold Ω embedded in it, on which the Laplace-Beltrami operator is defined. In this case the convergence (1) will be uniform on any Riemannian submanifold Ω' embedded in Ω whose distance from the boundary of Ω exceeds a positive–

* Of course, the subdomain Ω may coincide with the whole domain G.

** The domain D may coincide with Ω.

*** Recall that the system $\{u_k(x)\}$ is orthonormal with weight $\sqrt{g(x)}$ (see ([1])).

number (and if $G = \Omega$ is a compact manifold without boundary, then the convergence (1) will be uniform on the whole manifold Ω).

Combining the main theorem formulated above with two principal theorems of our work [2], we arrive at the conclusion that, for an arbitrary f.s.f. of the Beltrami operator in a subdomain Ω, we have established the following results.

1) For an arbitrary N-dimensional domain, final conditions, in the classes of S. L. Sobolev W_2^α, for the localization of the Fourier series of a finite function $f(x)$ have been established (it is proved that for $\alpha \geq (N-1)/2$ the localization principle is valid for the Fourier series; at the same time, for $\alpha < (N-1)/2$ this principle does not hold).

2) For an arbitrary domain of any odd number $N \geq 3$ of dimensions, final conditions, in the Hölder classes $C^{(n,\alpha)}$, for localization and uniform convergence of the Fourier series of a finite function $f(x)$ have been established (it is proved that the membership of $f(x)$ in the class $C^{((N-3)/2,\alpha)}$ for $\alpha = 1$ ensures uniform convergence of the Fourier series in a subdomain; at the same time, the membership of $f(x)$ in the same class for $\alpha < 1$ does not even ensure localization of the Fourier series).

3) For an arbitrary domain of any even number N of dimensions, conditions close to final, in the Hölder classes $C^{(n,\alpha)}$, for localization and uniform convergence of the Fourier series of a finite function $f(x)$ have been established (it is proved that the membership of $f(x)$ in the class $C^{((N-2)/2,\alpha)}$

for $\alpha > 1/2$ ensures uniform convergence of the Fourier series in a subdomain; at the same time, the membership of $f(x)$ in the same class for $\alpha < 1/2$ does not even ensure localization of the Fourier series).

It should be noted that the conditions for localization and uniform convergence obtained by us in the present work for an arbitrary f.s.f. of the Beltrami operator, and in [2] for an arbitrary f.s.f. of the Laplace operator, are new (and, in a certain sense, final) even for multiple trigonometric Fourier series and Fourier series in eigenfunctions of concrete boundary-value problems, although the questions of convergence and localization of the indicated series have been investigated by a number of mathematicians (L. Tonelli, E. C. Titchmarsh, S. Bochner, C. Minakshisundaram, E. Stein, B. M. Levitan, O. A. Ladyzhenskaya, and others).

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1. A. A. Arsen' ev, V. A. Il' in, *DAN*, **190**, No. 6 (1970).

2. V. A. Il' in, *UMN*, **23**, No. 2, 61 (1968).

Note: Figure translations are in progress. See original paper for figures.