ON THE CARDINALITY OF EXTREMALLY DISCONNECTED SPACES AND COMPLETE BOOLEAN ALGEBRAS

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Abstract

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D. A. VLADIMIROV, B. A. EFIMOV

ON THE CARDINALITY OF EXTREMALLY DISCONNECTED SPACES AND COMPLETE BOOLEAN ALGEBRAS

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It is well known that every cardinal is the cardinality of some bicompactum. This situation changes if we pass to subcategories of bicompacta. For example, as was shown in (2) under the assumption (GCH), the cardinality of every infinite dyadic bicompactum either is a power of two or is the sum of a countable number of smaller cardinals. Here it is proved that under the assumption (GCH) the cardinality of every infinite extremally disconnected bicompactum either is a power of two or is unattainable*. The principal method of proof of this theorem is a further development of the authors' ideas (1, 3) on embedding free Boolean algebras into complete Boolean algebras. All the notation and concepts occurring here may be found in (1-3).

1. By $\mathbf{F}$ we denote the contravariant functor acting from the category of zero-dimensional bicompacta to the category of Boolean algebras. If X is a zero-dimensional bicompactum, then $\mathbf{F}(X)$ is the Boolean algebra of its open-closed subsets; conversely, if $\mathfrak{X}$ is a Boolean algebra, then $\mathbf{F}^{-1}(\mathfrak{X})$ is the Stone space of the algebra $\mathfrak{X}$. We transfer the action of the functor $\mathbf{F}$ in the natural way to cardinal invariants of topological spaces. For example, if wX is the topological weight of the space X, then $\mathbf{F}(wX) = \operatorname{card}\mathbf{F}(X)$ is the cardinality of the Boolean algebra $\mathfrak{X} = \mathbf{F}(X)$; if cX is the Suslin number of the space X, then $\mathbf{F}(cX) = t\mathbf{F}(X)$ is the type of the Boolean algebra; finally, if $\chi(x, X)$ is the character of the point x in the space X, then $\mathbf{F}(\chi(x, X)) = \chi(\mathfrak{F}, \mathfrak{X})$ is the character of the ultrafilter $\mathfrak{F}$ in the algebra $\mathfrak{X}$, i.e. the minimum of the cardinalities of bases of the ultrafilter $\mathfrak{F}$ in $\mathfrak{X}$. Further, let $t\mathfrak{X}$ be the weight of the complete Boolean algebra $\mathfrak{X}$, i.e. $\min\{\operatorname{card} A, A \subset \mathfrak{X}, \bar{A} = \mathfrak{X}\}$, where the bar denotes the closure of the set A in the (o)-topology of $\mathfrak{X}$. This means that A completely generates $\mathfrak{X}$, or, what is the same thing, $\mathfrak{X}$ is the smallest regular subalgebra of $\mathfrak{X}$ containing A. Let X be an extremally disconnected bicompactum; then the cardinal invariant $vX = \mathbf{F}^{-1}(t\mathbf{F}(X))$ will be called the algebraic weight of X. We note that there exist extremally disconnected bicompacta of

arbitrarily high topological weight having countable algebraic weight ([4], [5]).

2. Let $\mathfrak{X}$ be a complete infinite Boolean algebra. We note that $t\mathfrak{X}$ is the least among those cardinal numbers $\mathfrak{m}$ which possess the following property: every nonempty set $E \subset \mathfrak{X}$ contains a subset $E' \subset E$ with the same bounds and such that $\operatorname{card} E' \leqslant \mathfrak{m}$. Consider an arbitrary set T of cardinality $t\mathfrak{X}$ and the upward directed system A of all its finite subsets. It is clear that $\operatorname{card} A = t\mathfrak{X}$. As is known ([1]), the sets of the algebra $\mathfrak{X}$ that are closed in the (o)-topology are precisely those and only those sets which contain the limits of all possible (o)-convergent generalized sequences of their elements.

Lemma 1. *Let Γ be a directed set, $\{x_\gamma, \gamma \in \Gamma\}$ a generalized sequence (o)-converging to some x. Then there exists a generalized sequence $\{x_\gamma, \alpha \in A\}$ with the same (o)-limit.*

Proof. Put

$$U_\gamma = \bigvee_{\delta > \gamma} |x - x_\delta|.$$

It is clear that $U_\gamma \downarrow 0$.

Choose in Γ a subset Γ' of cardinality not exceeding $t\mathfrak{X}$, for which will be $\bigwedge_{\gamma \in \Gamma'} U_\gamma = 0$. Let φ be some mapping of T onto Γ'. For each $a \in A$ there is an index γ_a such that $\gamma_a > \varphi(t)$ for every $t \in a$. Next put $\widetilde{U}_a = U_{\gamma_a}$, if $a \in A$. The inequalities

$$\widetilde{U}_a \leq \bigwedge_{t \in a} U_{\varphi(t)}$$

show that $\widetilde{U}_a \xrightarrow{(0)} 0$. Further, since $|x - x_a| \leq \widetilde{U}_a$, the sequence $\{x_a,\ a \in A\}$ (o)-converges to x. The lemma is proved.

Lemma 2. Let $E \subset X$. Then

$$\operatorname{card} \overline{E} \leq (\operatorname{card} E)^{tX}.$$

Proof. Denote by ζ the least cardinal whose cardinality is greater than tX. Form a transfinite sequence $\{E_\xi,\ \xi < \zeta\}$ of sets, in which $E_0 = E$ and each E_ξ consists of all possible (o)-limits of generalized sequences of the form $\{x_a,\ a \in A\}$, formed from elements of the set $\bigcup_{\eta < \xi} E_\eta$. Whatever the generalized sequence $\Lambda = \{x_a,\ a \in A\}$, consisting of elements of the set

$$\mathcal{E} = \bigcup_{\xi < \zeta} E_\xi,$$

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* A cardinal is called (weakly) unattainable if it is a limit cardinal and regular. (GCH) is the generalized continuum hypothesis.

by Lemma 1 the set of its terms has cardinality not exceeding tX, and, consequently,

$$\Lambda \subset \bigcup_{\xi<\xi_0} E_\xi$$

for some $\xi_0 < \zeta$, since ζ is a regular ordinal. Thus $\overline{E} = \mathcal{E}$. Estimate the cardinality of $\mathcal{E}$. Let $\mathfrak{n} = \operatorname{card} E$, $t = tX$, and let t^+ be the cardinal following t. Then $\operatorname{card} E_\xi \leq \mathfrak{n}^t$ for every $\xi < \zeta$. Indeed, this inequality is true for $\xi = 0$; if it is true for all $\xi < \xi_0 < \zeta$, then

$$\operatorname{card} \left(\bigcup_{\eta<\xi_0} E_\eta \right) \leq t\mathfrak{n}^t \leq \mathfrak{n}^t \mathfrak{n}^t = \mathfrak{n}^t$$

and next

$$\operatorname{card} E_{\xi_0} \leq \left[\operatorname{card} \left(\bigcup_{\eta<\xi_0} E_\eta \right) \right]^A \leq (\mathfrak{n}^t)^t = \mathfrak{n}^{t^2} = \mathfrak{n}^t.$$

Now it is clear that

$$\operatorname{card} \overline{E} \leq \operatorname{card} \mathcal{E} \leq t^+ \mathfrak{n}^t \leq 2^t \mathfrak{n}^t \leq \mathfrak{n}^t \mathfrak{n}^t = \mathfrak{n}^t.$$

The lemma is proved.

Lemma 3. Let X be a complete Boolean algebra; let τ^* be the supremum of the weights of the homogeneous components of this algebra; let $t = tX$ be its type. Then there exist:
a) an independent system of elements T, and b) a disjoint system D, such that $\operatorname{card} T = \tau^*$, $\operatorname{card} D \leq t$, and $X = \overline{X}\langle T, D\rangle$. In other words, the union $T \cup D$ topologically generates the algebra X.

Proof. Form a decomposition of X into disjoint components X_α, homogeneous in weight, having weights τ_α, $\alpha < \zeta$. Here ζ is some ordinal whose cardinality does not exceed t. It is clear that $\tau^* = \sup \tau_\alpha$. We may assume that the weights increase with the index: $\tau_\beta \geq \tau_\alpha$ for $\beta \geq \alpha$. Denote by E_α the set of all ordinals whose cardinalities are strictly less than τ_α. Thus $E_1 \subset E_2 \subset \cdots$ and $\operatorname{card} E_\alpha = \tau_\alpha$. For each pair of indices (α, β), $\alpha \geq \beta$, construct a mapping $\varphi_{\alpha\beta}$ from E_α into E_β, putting $\varphi_{\alpha\beta}(\eta) = \eta$ for $\eta \in E_\beta$ and $\varphi_{\alpha\beta}(\eta) = 1$ for $\eta \in E_\alpha - E_\beta$. We have, obviously, for $\alpha \geq \beta \geq \gamma$: $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma}\varphi_{\alpha\beta}$, and for every α, $\varphi_{\alpha\alpha}$ is the identity mapping of E_α onto itself. In each of the components X_α there is a completely generating independent system of elements of cardinality τ_α ((1), p. 262, Theorem 2). We have in mind here relative independence of these systems, that is, independence with respect to the corresponding component X_α, regarded as an independent Boolean algebra. Let θ_α be some bijection, fixed from now on, of E_α onto T_α. Form a set T, including in it all sums of the form $\bigvee x_\alpha$, $\alpha < \zeta$, where $x_\alpha \in T_\alpha$ and, for $\alpha \geq \beta$, let

$$x_\beta = \theta_\beta \varphi_{\alpha\beta} \theta_\alpha^{-1}(x_\alpha).$$

We shall show that T is an independent set of elements. Take a finite set $x^1, x^2, \dots, x^m \in T$, whose elements are distinct. Let $X^i = \bigvee x^i_\alpha,\ \alpha < \zeta$. Then for any pair of indices $i, j \le m$ there is $\alpha < \zeta$ such that $x^i_\alpha \ne x^j_\alpha$. Then also for all $\alpha' > \alpha$ one will have $x^i_{\alpha'} \ne x^j_{\alpha'}$ (otherwise

$$x^i_\alpha = \theta_\alpha \varphi_{\alpha'\alpha} \theta^{-1}_{\alpha'}(x^i_{\alpha'}) = \theta_\alpha \varphi_{\alpha'\alpha} \theta^{-1}_{\alpha'}(x^j_{\alpha'}) = x^j_\alpha$$

). It is clear now that there is $\alpha_0 < \zeta$ such that, for all $\alpha \ge \alpha_0$, any x^i_α, x^j_α $(i \ne j)$ are distinct. And then, since $x^i_\alpha \in T_\alpha$, and T_α is relatively independent, for any p one has

$$x^1 \wedge \dots \wedge x^p \wedge Cx^{p+1} \wedge \dots \wedge Cx^m \ge x^1_{\alpha_0} \wedge \dots \wedge x^p_{\alpha_0} \wedge Cx^{p+1}_{\alpha_0} \wedge \dots \wedge Cx^m_{\alpha_0} > 0.$$

We have thereby proved the independence of the system T. The disjoint ⋯

the system D, by putting $U_\alpha = \bigvee x_\alpha,\ x_\alpha \in \mathfrak{X}_\alpha$. Then $D = \{U_\alpha\}$. It is easy to see that $\mathfrak{X} = \mathfrak{X}\langle T, D\rangle$, $\operatorname{card} D \le t$. It remains to determine the cardinality of the system T. Let $y \in T_{\alpha_0}$. Putting $x_\alpha = \theta_\alpha \theta^{-1}_{\alpha_0}(y)$ for $\alpha \ge \alpha_0$ and $x_\alpha = \theta_\alpha \varphi_{\alpha\alpha_0} \theta^{-1}_{\alpha_0}(y)$ for $\alpha < \alpha_0$, we see that the element $\bigvee_{\alpha > \zeta} x_\alpha$ belongs to T. Therefore $\operatorname{card} T \ge \tau_{\alpha_0}$ for every α_0, and hence $\operatorname{card} T \ge t^*$. Conversely, to an arbitrary ordinal $\gamma \in \bigcup_{\alpha < \zeta} E_\alpha$ we associate the least number $\bar\alpha = \alpha(\gamma)$ for which $\gamma \in E_{\bar\alpha}$, and then put $x_\alpha = \theta_\alpha(1)$ for $\alpha < \bar\alpha$ and $x_\alpha = \theta_\alpha(\gamma)$ for $\alpha \ge \bar\alpha$. Finally, let $x = x(\gamma) = \bigvee_{\alpha > \zeta} x_\alpha$. It is easy to see that $x \in T$ and that the system T contains no other elements distinct from elements of the form $x(\gamma)$. Thus a surjection of the set $\bigcup_{\alpha < \zeta} E_\alpha$ onto T has been constructed and $\operatorname{card} T \le \operatorname{card} \bigcup_{\alpha < \zeta} E_\alpha = t^*$. The lemma is proved.

Lemma 4. Let E be a base of some proper ultrafilter $\mathfrak{F}$ of the complete Boolean algebra $\mathfrak{X}$. Then there exists a principal ideal $\mathfrak{A}$ which, considered as an element of $\mathfrak{X}$, belongs to $\mathfrak{F}$, and the set $\{e \wedge \{\mathfrak{A}\},\ e \in E\}$ completely generates $\mathfrak{A}$.

Proof. We note that there exists a component $\mathfrak{X}_0 \subset \mathfrak{X}$ which is saturated by $\overline{\mathfrak{X}\langle E\rangle}$. Otherwise, by Lemma 2, p. 246 (1), there is an element $z \in \mathfrak{X}$ such that for every $x \in \overline{\mathfrak{X}(E)}$ the inequalities $z \wedge x > 0$ and $Cz \wedge x > 0$ hold. In particular, these inequalities are true for every $x \in E$. Since $\mathfrak{F}$ is an ultrafilter, either $z \in \mathfrak{F}$, or $Cz \in \mathfrak{F}$. Let $z \in \mathfrak{F}$. Then, by the definition of a base E, there exists $x \in E$ such that $0 < x < z$. But in this case $Cz \wedge x = 0$. A contradiction. Thus $\overline{\mathfrak{X}(E)}$ saturates some component $\mathfrak{X}_0 \subset \mathfrak{X}$. Let $y = \{\mathfrak{X}_\alpha,\ \alpha \in A\}$ be a maximal family of pairwise disjoint components, each of which is saturated by $\overline{\mathfrak{X}\langle E\rangle}$. Then $x = \{\mathfrak{A}\} = \bigvee y,\ y \in \bigcup_{\alpha \in A} \mathfrak{X}_\alpha$, is a principal ideal of $\mathfrak{X}$, and $\overline{\mathfrak{X}\langle E\rangle}$ saturates $\mathfrak{A}$. We prove that $x \in \mathfrak{F}$. If $x \notin \mathfrak{F}$, then there exists a component $\mathfrak{X}_0 \subset Cx \in \mathfrak{F}$ which is saturated by $\overline{\mathfrak{X}\langle E\rangle}$, contrary to the maximality of η. The lemma is proved.

3. Let $f : X \to Y$ be a continuous mapping of the topological space X onto the space Y. We shall call the mapping f **thin** if $\operatorname{int} f^{-1}(x) = \emptyset$ for every nonisolated point $x \in Y$.

Lemma 5. Let $\varphi : E \to \mathfrak{X}$ be the identical embedding of the subalgebra E into the algebra $\mathfrak{X}$ such that E completely generates $\mathfrak{X}$. Then the mapping $\mathrm{F}^{-1}(\varphi) : \mathrm{F}^{-1}(\mathfrak{X}) \to \mathrm{F}^{-1}(E)$ is a thin mapping.

Proof. Suppose the contrary. Let x be some nonisolated point of $\mathrm{F}^{-1}(E)$ such that $U = \mathrm{int}[\mathrm{F}^{-1}(\varphi)]^{-1}(x) \neq \emptyset$. Since $\mathrm{F}^{-1}(x)$ is extremally disconnected, U is open-and-closed. Let V be another open-and-closed set, with $V \subset U$ and $U - V = V_0 \neq \emptyset$. Denote by $a = \mathrm{F}(V)$ and $a_0 = \mathrm{F}(V_0)$ the corresponding elements of the Boolean algebra $\mathfrak{X}$. We note that for every open-and-closed $W \subset \mathrm{F}^{-1}(E)$ the symmetric difference $|W - V| \supset V_0$. This means that for every element $x \in E$ we have $|x - a| > a_0$, which contradicts the given condition ($\bar{E} = \mathfrak{X}$). The lemma is proved.

4. **Main results.** Let t be a cardinal. Denote by $\mathrm{Ex}(t)$ the function of t which is equal to $2^t = \exp t$ if t is attainable, or to $\sum_{\sigma<t} \exp(\sigma)$ if t is unattainable. As Erdős and Tarski showed (^{6}), if the type of the Boolean algebra $\mathfrak{X}$ is equal to t, and t is attainable, then in $\mathfrak{X}$ there exists a family of disjoint elements of cardinality t. We note that, by the results of Kantorovich–Fichtenholz–Hausdorff (^{7}) on independence, every complete Boolean algebra $\mathfrak{X}$ containing t disjoint elements necessarily contains a free subalgebra of cardinality $\exp t$. Combining this observation with Lemmas 2 and 3, we obtain the following two main theorems.

Theorem 1. The cardinality of an infinite complete Boolean algebra $\mathfrak{X}$ satisfies the following inequality

$$\max[\mathrm{Ex}(t), \tau^*] \leq \mathrm{card}\, \mathfrak{X} \leq \min[\tau^t, (\tau^* + t)^t], \tag{*}$$

where t is the type of the algebra $\mathfrak{X}$, τ is the weight of $\mathfrak{X}$, and τ^* is the supremum of the weights of homogeneous components. Moreover, $\mathfrak{X}$ contains a free subalgebra of cardinality $\tau^* + \exp(t)$, if t is attainable, or $\tau^* + \xi$ for every $\xi < \mathrm{Ex}(t)$, if t is unattainable.

Theorem 2. The weight of an infinite extremally disconnected bicompact space X satisfies the inequality $(*)$, where t is the Suslin number of X, τ is the algebraic weight of X, and τ^* is the supremum of the algebraic weights of the open-closed subsets of X that are homogeneous with respect to this weight. Moreover, X is continuously mapped onto a generalized Cantor discontinuum $D^{\mathfrak{m}}$, where $\mathfrak{m} = \tau^* + \exp(t)$, if t is attainable, or $\mathfrak{m} = \tau^* + \xi$ for every $\xi < \mathrm{Ex}(t)$, if t is unattainable.

Below we give a table in which the values of the left- and right-hand sides of the inequality $(*)$ are computed as functions of t. Let us note that always $\tau^* \leq \tau$. Here t-a denotes that t is an attainable cardinal, and t-u denotes an unattainable cardinal.

t	t	$\max[\mathrm{Ex}(t), \tau^*]$	$\min[\tau^t, (\tau^* + t)^t]$	No.
$t < \tau^*$	t-a	$\tau^* + \exp(t)$	$(\tau^*)^t$	1
$t < \tau^*$	t-u	$\tau^* + \sum_{\sigma<t} \exp(\sigma)$	$(\tau^*)^t$	2
$\tau^* \leq t$	t-a	$(\tau^*)^t = \exp(t)$	$(\tau^*)^t = \exp(t)$	3
$\tau^* \leq t$	t-u	$\sum_{\sigma<t} \exp(\sigma)$	$(\tau^*)^t = \exp(t)$	4

Theorem 3 (GCH). The cardinality of an infinite extremally disconnected bicompact space X is either equal to a power of two or is unattainable.

Proof. Let us note that for bicompact spaces one has

$$wX \leq \operatorname{card} X \leq \exp(wX). \tag{**}$$

In cases 1) and 2), listed in the table, we have $\tau^* \leq wX \leq (\tau^*)^t$, with $t < \tau^*$, and X is mapped onto D^{τ^*}. Hence, applying $(**)$, we obtain

$$\exp(\tau^*) \leq \operatorname{card} X \leq \exp[(\tau^*)^t] \leq \exp\exp\tau^*.$$

Thus, under (GCH), either $\operatorname{card} X = \exp\tau^*$ or $\operatorname{card} X = \exp\exp\tau^*$. In case 3) one has $wX = \exp t$, and X is mapped onto $D^{\exp t}$. Hence $\operatorname{card} X = \exp\exp t$. Finally, in case 4) we have $t \leq wX \leq \exp t$, with t unattainable. Hence, again applying $(**)$, we obtain

$$t \leq \operatorname{card} X \leq \exp\exp t.$$

Thus, under (GCH), the cardinality of X is equal either to t, or to $\exp t$, or to $\exp\exp t$, with t unattainable. The theorem is proved.

Theorem 4. Let X be an extremally disconnected bicompact space and let $x \in X$. Then there exists a neighborhood U of the point x such that $wU \leq [\chi(x, X)]^{c^x}$.

Theorem 5. Every extremally disconnected bicompact space homogeneous with respect to algebraic weight τ is mapped onto D^τ by means of an irreducible mapping.

Leningrad State University named after A. A. Zhdanov
Central Economics and Mathematics Institute
of the Academy of Sciences of the USSR, Moscow

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Note: Figure translations are in progress. See original paper for figures.

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