ON THE CONVERGENCE OF THE METHOD OF "ELASTIC" SOLUTIONS IN NONLINEAR VISCOELASTICITY

THEORY OF ELASTICITY

1970

SovietRxiv

Abstract

Full Text

UDC 539.374 + 517.93

THEORY OF ELASTICITY

B. E. POBEDRYA

ON THE CONVERGENCE OF THE METHOD OF "ELASTIC" SOLUTIONS IN NONLINEAR VISCOELASTICITY

(Presented by Academician L. I. Sedov, May 29, 1970)

1. Consider the principal quasi-linear viscoelastic incompressible medium in which the relation between the deviators of stresses s_{ij} and strains e_{ij} has the form ([1])

$$s_{ij} = \int_0^t \Gamma(t-\tau)e_{ij}(\tau)\,d\tau - \int_0^t \Gamma_\varphi(t-\tau)\varphi(e)e_{ij}(\tau)\,d\tau, \tag{1}$$

where $e \equiv e_{ij}(\tau)e_{ij}(\tau) = e_u^2(\tau)$.

The linear relaxation kernel $\Gamma(t)$ and the nonlinear one $\Gamma_\varphi(t)$ are decomposed into singular and regular components

$$\Gamma(t) = \dot{\Gamma}\delta(t) + \tilde{\Gamma}(t), \qquad \Gamma_\varphi(t) = \dot{\Gamma}_\varphi\delta(t) + \tilde{\Gamma}_\varphi(t).$$

If $\dot{\Gamma}_\varphi = 0$, then the corresponding theory is called the principal quasi-linear theory with instantaneous linear elasticity ([1]). We write relation (1) briefly in the form

$$s_{ij} = \dot{\Gamma}e_{ij} - \dot{\Gamma}_\varphi\varphi(e)e_{ij}. \tag{2}$$

We shall regard the strains as small,

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad e_{ii} = 0. \tag{3}$$

Substituting relations (3) into (2), we obtain

$$s_{ij}(\mathbf{u}) = s^0_{ij}(\mathbf{u}) + s^1_{ij}(\mathbf{u}), \qquad s^0_{ij}(\mathbf{u}) = \dot{\Gamma}e_{ij}(\mathbf{u}) \equiv \frac{1}{2}\dot{\Gamma}(u_{i,j} + u_{j,i}),$$

$$s^1_{ij}(\mathbf{u}) = -\frac{1}{2}\dot{\Gamma}_\varphi \varphi(e)(u_{i,j} + u_{j,i}), \qquad e = \frac{1}{2}(u_{i,j}u_{i,j} + u_{i,j}u_{j,i}).$$

The quasistatic problem of the viscoelasticity theory under consideration consists in integrating the three equilibrium equations

$$s_{ij,j}(\mathbf{u}) + P_{,i} + \rho F_i = 0$$

with respect to the displacement vector $\mathbf{u}$, subject to the boundary conditions

$$\mathbf{u}\big|_\Sigma = \mathbf{u}^0 \tag{4}$$

or

$$\sigma_{ij} l_j\big|_\Sigma = S^0_i. \tag{5}$$

Here, if on the boundary Σ bounding the volume V under consideration the conditions (4) are prescribed, one says that the first boundary-value problem is being solved, while if the conditions (5) are prescribed, the second boundary-value problem is being solved. Here $\sigma_{ij} = s_{ij} + P\delta_{ij}$, P is a function determined as a result of solving the problem, and ρF_i are body forces.

We note that, by changing the body forces $\rho\mathbf{F}$, the boundary conditions (4) can be made homogeneous,

$$\mathbf{u}\big|_\Sigma = 0. \tag{6}$$

Therefore, by the first boundary-value problem we shall henceforth mean the problem with boundary conditions (6).

Denote

$$P_{,i} + \rho F_i \equiv -f_i(\mathbf{x}, t).$$

Following ([2]), we shall call a vector-function $\mathbf{u}$ a generalized solution of the first and second boundary-value problems, respectively, if it satisfies the following integral identities for any continuously differentiable vector-function $\mathbf{v}$:

$$\int_V \left[s_{ij}(\mathbf{u})\, e_{ij}(\mathbf{v}) + f_i v_i\right]\, dV = 0,$$

$$\int_V \left[s_{ij}(\mathbf{u})\, e_{ij}(\mathbf{v}) + f_i v_i\right]\, dV + \int_\Sigma S^0_i v_i \, d\Sigma.$$

Introduce a scalar product for certain differentiable functions by the formula

$$(\mathbf{u}, \mathbf{v}) = \int_V e_{ij}(\mathbf{u})\, e_{ij}(\mathbf{v})\, dV. \tag{7}$$

We shall define the solution of the posed problems in the space H, obtained by completion in the norm (7) of the set of twice continuously differentiable vector-functions satisfying, in the case of the first boundary-value problem, condition (6). As is known ([2,3]), the condition of boundedness in H of the integrals

$$\int_V f_i v_i\, dV \quad \text{and} \quad \int_\Sigma S_i^0 v_i\, d\Sigma$$

holds when

$$f_i \in L_p(V), \quad p > 6/5; \qquad S_i^0 \in L_q(\Sigma), \quad q > 4/3.$$

Suppose now that the integral equation

$$x = \int_0^t \Gamma(t-\tau)\, y(\tau)\, d\tau$$

is uniquely solvable in the form

$$y = \int_0^t K(t-\tau)\, x(\tau)\, d\tau,$$

so that ([1])

$$\int_0^t K(t-\tau)\Gamma(\tau)\, d\tau = \delta(t). \tag{8}$$

Let, further, the function $\varphi(e)$ be such that for any $t \geq 0$

$$0 \leq \varphi(e) \leq \varphi(e) + \frac{\varphi(e) - \varphi(e')}{e - e'}\, e' \leq \eta. \tag{9}$$

In addition, the kernel $\Gamma_\varphi(t)$, for any $t \geq 0$, satisfies

$$|\Gamma_\varphi(t)| \leq A\Gamma(t). \tag{10}$$

The method of "elastic" ([4]) solutions consists in the successive solution of problems of the linear theory of viscoelasticity

$$s^0_{ij,j}(\mathbf{u}_{(0)}) = f_i,$$

$$s^0_{ij,j}(\mathbf{u}_{(n)}) = -s^1_{ij,j}(\mathbf{u}_{(n-1)}), \qquad n \geq 1,$$

with, for example, conditions (6) imposed. Here the unknown function P is determined from the solution of the linear problem when determining the zeroth approximation.

Theorem. *Under conditions (8), (9), (10), the method of "elastic" solutions converges to the unique generalized solution*, if*

$$A\eta \equiv q < 1. \tag{11}$$

* The generalized solution will be classical if it is twice continuously differentiable with respect to $\mathbf{x}$.

and the initial approximation $\mathbf{u}_{(0)}$ is such that

$$\|\mathbf{u}_{(0)}\| = \left\{\int_V [e_{u(0)}]^2 dV\right\}^{1/2} \leq \frac{1-q}{q}\, r, \tag{12}$$

where r is a positive number for which the inequality

$$r \geq \|\mathbf{u} - \mathbf{u}_{(0)}\|. \tag{13}$$

holds.

Proof We shall carry it out for the first boundary-value problem, since the proof for the second boundary-value problem is analogous.

Consider the identity

$$\int_V e_{ij}(\mathbf{u}) e_{ij}(\mathbf{v})\, dV = \int_V [\tilde{K}\tilde{\Gamma}_\varphi \varphi(e) e_{ij}(\mathbf{u})] e_{ij}(\mathbf{v}) dV - \int_V [\tilde{K} f_i] v_i\, dV. \tag{14}$$

On the left-hand side of relation (14) stands the scalar product in the space H: $(\mathbf{u}, \mathbf{v})$. The right-hand side of (14), on the basis of (9) and (10), is a linear functional with respect to $\mathbf{v}$ in this space. By the Riesz theorem this functional can be represented in the form of the scalar product $(\mathbf{u}^*, \mathbf{v})$, where $\mathbf{u}^* \in H$ [5]. In other words, an operator Q is defined in H, which to each vector-function $\mathbf{u}$ assigns the vector-function $\mathbf{u}^*$. Consequently, the question of finding the generalized solution is reduced to solving the operator equation

$$\mathbf{u} = Q\mathbf{u}.$$

We shall prove that the operator Q is a contraction operator; then, according to the contraction mapping principle, from this will follow the convergence of the method of "elastic" solutions and the existence and uniqueness of the solution. (If, of course, the problem of the linear theory of viscoelasticity has a unique solution [6].)

It follows from (9) and (10) that for any $\mathbf{u}_1$ and $\mathbf{u}_2$

$$|(Q\mathbf{u}_1 - Q\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)| \leq q\|\mathbf{u}_1 - \mathbf{u}_2\|^2,$$

whence

$$\|Q\mathbf{u}_1 - Q\mathbf{u}_2\| \leq q\|\mathbf{u}_1 - \mathbf{u}_2\|.$$

Further, from (12) and (13) we have

$$\|Q\mathbf{u} - \mathbf{u}_{(0)}\| = \|Q\mathbf{u} - Q\mathbf{u}_{(0)}\| + \|Q\mathbf{u}_{(0)} - \mathbf{u}_{(0)}\| \leq rq + q\frac{1-q}{q}r = r.$$

It follows from this that, for $A\eta \equiv q < 1$, the operator Q is a contraction operator, and the theorem is proved.

Let us note that if the function $\varphi(e)$ is such that $\varphi(e_{(0)}) = 0$ (i.e., there exists some small "region of linearity" in which $\varphi(e) = 0$), then in this case $\eta = 0$, and the method considered, as follows from (12), converges for any zero approximation. In the opposite case (when the "region of linearity" does not exist), and it is known to us that for the solution $\mathbf{u}$: $e_u \leq M$, i.e. $\|\mathbf{u}\| \leq M\sqrt{V}$, then in (12) and (13) one should put $r = A\eta M\sqrt{V}$, and then the indicated iterative process will converge if the zero approximation $\mathbf{u}_{(0)}$ satisfies the condition

$$e_{u(0)} \leq (1 - A\eta)M.$$

2. Let us now consider a certain generalization of the method of "elastic" solutions. We have the identity:

$$\int_V e_{ij}(\mathbf{u})e_{ij}(\mathbf{v})\,dV = \int_V e_{ij}(\mathbf{u})e_{ij}(\mathbf{v})\,dV - \beta\left\{\int_V e_{ij}(\mathbf{u})e_{ij}(\mathbf{v})\,dV - \int_V [\tilde{K}\tilde{\Gamma}_\varphi \varphi(e)e(\mathbf{u})]e_{ij}(\mathbf{v})\,dV + \int_V [\tilde{K}f_i]v_i\,dV\right\}. \tag{15}$$

For $\beta = 1$ this identity coincides with (14). Repeating all the arguments of Sec. 1, we obtain for (15)

$$\|Qu_1 - Qu_2\| \leq \psi(\beta)\|u_1 - u_2\|; \qquad \psi(\beta) \equiv 1 - \beta(1-q),$$

where β varies in the interval $0 < \beta \leq 2/(2-q)$; moreover, the function $\psi(\beta) < 1$ attains its least value $\psi(\beta) = q/(2-q)$ for $\beta = 2/(2-q)$. Obviously, on the interval $0 < q < 1$ the inequality

$$0 < q/(2-q) < q.$$

holds.

Therefore the iterative process constructed according to scheme (15), with $\beta = 2/(2-q)$, converges faster than in the case of (14). Thus the theorem proved in Sec. 1 is also valid here, with the inequality (12) replaced by

$$\|u^{(0)}\| \leq \frac{1-\psi(\beta)}{\psi(\beta)}\, r = \frac{\beta(1-q)}{1-\beta(1-q)}\, r. \tag{16}$$

For $\beta = 2/(2-q)$, from (16) we have:

$$\|u^{(0)}\| \leq 2r(1-q)/q,$$

i.e., the domain of admissible initial approximations is doubled.

Moscow State University
named after M. V. Lomonosov

Received
28 V 1970

CITED LITERATURE

1. A. A. Ilyushin, B. E. Pobedrya, *Fundamentals of the Mathematical Theory of Thermo-Viscoelasticity*, Nauka, 1970.
2. I. I. Vorovich, Yu. P. Krasovskii, DAN, 126, No. 4 (1959).
3. D. L. Bykov, V. A. Shachnev, PMM, 33, No. 2 (1968).
4. A. A. Ilyushin, *Plasticity*, 1948.
5. L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, Nauka, 1965.
6. B. E. Pobedrya, DAN, 190, No. 2 (1970).

Note: Figure translations are in progress. See original paper for figures.