SOME QUESTIONS IN THE THEORY OF EXTREMAL PROBLEMS FOR UNIVALENT FUNCTIONS OF THE CLASS \(S\)

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Abstract

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SOME QUESTIONS IN THE THEORY OF EXTREMAL PROBLEMS FOR UNIVALENT FUNCTIONS OF THE CLASS S

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1. We shall use the notation and definitions introduced by us in our preceding note (4). Let $f \in S_n$, and let $\delta f(z)$ be its first variation; put, as before, $\xi(w) = \delta f(z)$, $z = f^{-1}(w)$. Denote the totality of functions $\xi(w)$ by $\mathcal{H}_0 = \mathcal{H}_0(f)$. Let $\mathcal{T}$ be the boundary tree of the function f, and $G(\mathcal{T})$ the complement of $\mathcal{T}$ in the whole w-sphere. We note that the function $\xi(w) \in \mathcal{H}_0$ is regular in $G(\mathcal{T})$ and satisfies the conditions $\xi(0) = \xi'(0) = 0$ and

$$\operatorname{Im}\{Q(w)/w^2[\xi_+(w) - \xi_-(w)]\} = 0. \tag{1}$$

It is clear that $\mathcal{H}_0$ is a linear manifold over the field of real numbers. On $\mathcal{H}_0$ consider the bilinear form

$$\operatorname{Re}\left\{\frac{1}{\pi}\iint_{|w|\geq r_0} \frac{d\xi^*}{dw}\overline{\frac{d\eta^*}{dw}}\,d\sigma - \frac{1}{2\pi i}\int_{|w|=r_0} \xi^*\,\overline{d\eta^*} + \vartheta\sum_1^n \xi_k\overline{\eta_k}\right\}, \tag{2}$$

where $d\sigma$ is an element of area, r_0 is an arbitrary constant $\leq 1/4$, and ϑ is a certain constant, chosen so that for $\xi = \eta \neq 0$ expression (2) is strictly greater than zero. It can be shown that if $Q(w) = \sum_1^n A_k w^{-k}$ is normalized by the condition $\sum_1^n |A_k|^2 = 1$, then ϑ can be taken in the form $\vartheta = \alpha_0^n$, where α_0 is an absolute constant. The form (2) is symmetric and positive; denote it by (ξ, η) and put $\|\xi\| = (\xi, \xi)^{1/2}$. Completing $\mathcal{H}_0$ in the norm introduced, we obtain a real Hilbert space $\mathcal{H}$. The elements of $\mathcal{H}$ are naturally to be regarded as first variations of the function $f(z)$.

With the function $f \in S_n$ one can associate one more Hilbert space. Denote by $\mathcal{E}$ the totality of functions regular in $G(\mathcal{T})$, normalized by the conditions $\xi(0) = \xi'(0) = 0$, $\xi \in \mathcal{E}$, and such that $\|\xi\| < \infty$. If $\xi \in \mathcal{E}$, then almost everywhere on $\mathcal{T}$ it has finite limiting values $\xi_+(w)$, $\xi_-(w)$. Denote by $\mathcal{D}$ the set of those elements of $\mathcal{E}$ for which condition (1) is satisfied almost everywhere on $\mathcal{T}$. It is clear that $\mathcal{H} \subseteq \mathcal{D}$. It can be shown that $\mathcal{D} = \mathcal{H} \oplus R$, where R is a finite-dimensional subspace whose elements are certain rational fractions. In a number of questions it is important to know when R is empty.

Taking into account formula (19) of note (^{4}), we shall write the second variation of the functional $L(f)$ in the form $\delta^2 L(f;\xi)$. If $\xi \in \mathcal{H}_0$, then $\delta^2 L(f;\xi)$ and the norm $\|\xi\|$ are related by

$$\delta^2 L(f;\xi) = -\|\xi\|^2 + \vartheta \sum_1^n |\xi_k|^2 - \frac{1}{2} \operatorname{Re} \left\{ \sum_{k=1}^{n-1} \xi_k \sum_{l=1}^{n-k} (k+l+2) A_{k+l} \xi_l \right\}. \qquad (2^*)$$

2. The study of extremal problems on the compactum S for functionals of the form $J(f) = F(a_1, \dots, a_n)$ is most closely connected with the study of the structure of the coefficient region of the class S. Let R^{2n} be the real

Euclidean space; points of R^{2n} will be written in complex form, $x \in R^{2n}$, $x = (x_1, \dots, x_n)$, or $x = (a_1, \dots, a_n)$. The coefficient domain $\mathcal{V}_n$ is the image of the compact set S under the mapping $F : S \to R^{2n}$ such that the image of $f \in S$ is the point $x = (a_1, \dots, a_n) \in \mathcal{V}_n$. The preimage $x \in \mathcal{V}_n$ will be denoted by S_x. The structure of $\mathcal{V}_n$ was investigated by Schaeffer and Spencer in (^{1}). They showed that $\mathcal{V}_n$ is homeomorphic to the $2n$-dimensional ball $|x| \leq 1$, and established that $x \in \operatorname{Int} \mathcal{V}_n$ if and only if S_x contains a bounded function. Schaeffer and Spencer established that if $x \in \partial \mathcal{V}_n$, then $S_x \cap S_n \neq \emptyset$. From this theorem of Schaeffer and Spencer and Teichmüller' s theorem (^{2}) it follows that for $x \in \partial \mathcal{V}_n$, S_x consists of a single function. Thus, from the results of Schaeffer, Spencer, and Teichmüller it follows that S_n and $\partial \mathcal{V}_n$ are homeomorphic, i.e., S_n is homeomorphic to the $(2n-1)$-dimensional sphere.

For the theory of extremal problems the metric properties of the boundary $\mathcal{V}_n$ are of great importance. These questions are rather difficult, and works in which the smoothness of the boundary of the coefficient domain is studied are almost nonexistent. An exception is the work (^{3}), in which the authors announced a result stating that at the points of a certain set lying on $\partial \mathcal{V}_n$ there exists a tangent hyperplane.

Since S_n and $\partial \mathcal{V}_n$ are homeomorphic, at each point $x \in \partial \mathcal{V}_n$ an extremal function $f(z) = f(z;x)$ is defined, and hence also the corresponding cone $K_e(f) = K_x$ of vectors associated with the function $f(z,x)$. Denote by R the set of those $x \in \partial \mathcal{V}_n$ for which $\dim \Lambda_x = 1$. It is not difficult to show that R is an open subset of $\partial \mathcal{V}_n$. Let the complement of R in $\partial \mathcal{V}_n$ be denoted by $\mathcal{A}$. Thus, $\partial \mathcal{V}_n = R \cup \mathcal{A}$, and if $x \in \mathcal{A}$, then $\dim \Lambda_x > 1$. In a neighborhood of points $x^0 \in R$ the structure of the boundary $\mathcal{V}_n$ is rather simple. One can show that in a new system of rectangular coordinates (y, τ), $y = (y_1, \dots, y_{2n-1})$, with origin at x^0 and with the τ-axis directed along the vector $\lambda \in K_{x^0}$, the boundary $\mathcal{V}_n$ in a neighborhood of x^0 will be the graph of a continuous function

$\tau = \psi(y) = \psi(y_1, \dots, y_{2n-1})$, $\psi(0) = 0$. For $|y| \le \rho$ the function $\psi(y)$ is continuously differentiable and $\operatorname{grad}\psi(0) = 0$. It is not difficult to establish a connection between the differential properties of $\psi(y)$ at a certain point $x \in \partial V_n$ and the second variation of the linear functional determined by the vector $\lambda \in K_x$. In this way one can show that $\operatorname{grad}\psi(y)$ satisfies a Lipschitz condition for $|y| \le \rho$, and find the set of points from R where the second differential of the function $\psi(y)$ does not exist. See below for more details on this.

3. Let $f_0 \in S_n$, $\lambda^0 \in K_e(f_0)$, $|\lambda^0| = 1$; define, by the vector λ^0, the linear functional
$$L(f) = \operatorname{Re}\sum_1^n \overline{\lambda_k^0} a_k.$$
Fix arbitrary quantities $t_1, \dots, t_n$ subject to the condition
$$\operatorname{Re}\sum_1^n \overline{\lambda_k^0} t_k = 0,$$
and consider the set of first variations of the function $f_0(z)$ such that $\delta a_k = t_k$, $k = 1, \dots, n$. Considering these first variations as functions of w, we obtain a certain subset $\mathfrak{Z}_0$ of the linear manifold $\mathfrak{H}_0$; moreover, if $\xi(w) \in \mathfrak{Z}_0$,
$$\xi(w) = \sum_1^\infty \xi_k w^{k+1},$$
then $\xi(w)$ will have fixed first n coefficients $\xi_k = \mathfrak{z}_k$, $k = 1, \dots, n$. The quantities $\mathfrak{z}_1, \dots, \mathfrak{z}_n$ are readily computed from the quantities $t_1, \dots, t_n$. Consider the problem of determining
$$\sup_{\xi \in \mathfrak{Z}_0} \delta^2 L(f; \xi). \tag{3}$$
From formula (2) we obtain that this problem is equivalent to the problem of determining
$$\inf_{\xi \in \mathfrak{Z}} \|\xi\|. \tag{4}$$

Let $\mathfrak{Z}$ be the closure of $\mathfrak{Z}_0$; it is clear that the lower bound in (4) will not change if it is computed over the set $\mathfrak{Z}$. The set $\mathfrak{Z}$ is flat, and therefore, as is known, there is an element $\zeta \in \mathfrak{Z}$ such that
$$\inf_{\xi \in \mathfrak{Z}} \|\xi\| = \|\zeta\|.$$

The function $\zeta(w)$ will satisfy a differential equation, and in order to obtain it, it is useful to make use of the relation between the quantities (3) and (4) and formula (17) of our note [4]. As a result of the computations we obtain the equation for $\zeta(w)$
$$\sqrt{\frac{\overline{Q(w)}}{w^2}}\,\frac{d}{dw}\left[\sqrt{\frac{\overline{Q(w)}}{w^2}}\,\zeta(w)\right] + w^{-2}\widetilde{Q}(w)\cdot\frac{\widetilde{P}(z)}{\left(Df_0(z)\right)^2} = 0, \qquad z = f_0^{-1}(w), \tag{5}$$

where $\widetilde{Q}(w)$ is a certain polynomial in $1/w$ of degree not exceeding n, and

$$\widetilde{P}(z) = \sum_{-n}^{n} \widetilde{B}_k z^k, \qquad \operatorname{Im} \widetilde{P}(z) = 0 \quad \text{for } |z| = 1.$$

Equation (5) is the variational equation for equation (9) (see [4]), which is satisfied by the extremal function $f_0(z)$. Thus the theory presented here is, as it were, an analogue of Jacobi' s theory in the classical calculus of variations.

When $\mathscr{H}$ is a proper subspace of the space $\mathfrak{H}$, one may consider the subset $\mathfrak{Z}^* \subset \mathfrak{H}$ of those $\xi \in \mathfrak{H}$,

$$\xi(w) = \sum_{1}^{\infty} \xi_k w^{k+1},$$

for which $\xi_k = 0,\ k = 1, \dots, n$. Analogously to (4), introduce the quantity

$$\inf_{\xi \in \mathfrak{Z}^*} \|\xi\|. \tag{6}$$

Everything said above about problem (4) also holds in the present case. Denote by $D[f^0; t]$ the upper bound (3). Taking (2^*) into account, we see that $D(f^0; t)$ is connected by a rather simple relation with (4). By means of an identical relation, using the quantity (6), we define the functional $\mathfrak{D}[f^0; t]$. It is not hard to show that $D[f^0; t]$ and $\mathfrak{D}[f^0; t]$ are quadratic forms in the elements of the vector t. In view of the homeomorphism between $\partial \mathscr{V}_n$ and S_n, we obtain at each point of the set R the quadratic forms $D[x; t] = D[f; t]$, $f = f(z; x)$, and $\mathfrak{D}[x; t] = \mathfrak{D}[f; t]$.

4. Let us return to the constructions of item 2 and consider a local coordinate system (y, τ). Denote by $\mathfrak{B}$ the graph of the function $\tau = \psi(y)$ on the disk

$$\mathfrak{U} = \{y : \ |y| \le \rho\}.$$

Denote by $\mathfrak{Y}$ the collection of tangent vectors to the set $\mathfrak{B}$. The set $\mathfrak{Y}$ may be identified with the set of pairs (y, v), $y \in \mathfrak{U}$, while v is an arbitrary vector from R^{2n-1}. By the pair (y, v) we uniquely determine the point $(y, \psi(y)) \in \mathfrak{B}$ and the vector t, lying in the tangent hyperplane to $\mathfrak{B}$ at the point $(y, \psi(y))$. Thus $D[x; t]$ and $\mathfrak{D}[x; t]$ become functions of a point of the set $\mathfrak{Y}$. For convenience put

$$\widetilde{D}[y; v] = \left(|\operatorname{grad} \psi(y)|^2 + 1\right)^{1/2} D[x; t]$$

and, respectively,

$$\mathfrak{D}[y; v] = \left(|\operatorname{grad} \psi(y)|^2 + 1\right)^{1/2} \mathfrak{D}[x; t].$$

Define the second lower derivative number of Vallee-Poussin of the function $\psi(y)$

$$\underline{D}_v^2 \psi(y) = \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} \left[\psi(y + \varepsilon v) - \psi(y) - \varepsilon(\operatorname{grad} \psi(y), v)\right].$$

The upper derivative number is defined analogously.

The connection between the theory of the second variation and the structure of the boundary $\mathcal{V}_n$ is given by the simple inequality

$$\widetilde{D}[y;v] \leq \underline{D}_v^2\psi(y). \tag{7}$$

It can be shown that in (7) the equality sign is attained almost everywhere on the disk $\mathfrak{U}$. In this way we also obtain the aforementioned theorem stating that grad $\psi(y)$ satisfies a Lipschitz condition for $|y| \leq \rho$. It turns out that, in order that a second differential $\psi(y)$ exist at the point y, it is sufficient that $\mathcal{H}_y = \mathfrak{h}_y$, where $\mathcal{H}_y$ and $\mathfrak{h}_y$ are Hilbert spaces of first variations determined by the corresponding extremal function.

5. We have seen that the question of when the Hilbert spaces $\mathcal{H}$ and $\mathfrak{h}$ coincide is essential. The answer to this question is given in terms of the properties of the boundary tree $\mathcal{T}$ of the extremal $f(z)$, or, more precisely, the answer depends on the relationship between $\mathcal{T}$ and the graph of $\mathcal{U}$.

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Note: Figure translations are in progress. See original paper for figures.