ON BICOMPACTA WITH NONCOINCIDING INDUCTIVE DIMENSIONS

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ON BICOMPACTA WITH NONCOINCIDING INDUCTIVE DIMENSIONS

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In ([2]) a description was given of a bicompactum X with $\dim X = \operatorname{ind} X = 2$, $\operatorname{Ind} X = 3$. Recently I. K. Lifanov and B. A. Pasynkov substantially simplified this example, constructing a bicompactum X with $\dim X = \operatorname{ind} X = 3$, $\operatorname{Ind} X = 4$*. In this note we shall construct a series of examples of this sort, which in one respect or another are better than those named.

§ 1. **A bicompactum R with $\dim R = \operatorname{ind} R = 2$, $\operatorname{Ind} R = 3$.** In the set $C_0 = I \times I \times D$, where $I = [0,1]$, $D = \{0,1\}$, introduce the lexicographic order as follows: $(r_1, r_2, r_3) > (r_1', r_2', r_3')$ if $r_1 > r_1'$, or $r_1 = r_1'$, $r_2 > r_2'$, or $r_1 = r_1'$, $r_2 = r_2'$, $r_3 > r_3'$. An order is introduced analogously in the product $J = I \times K$, where K is the Cantor perfect set. Both these spaces, as is easily seen, are ordered zero-dimensional bicompacta with the first axiom of countability. In what follows we shall use the bicompactum constructed in ([3]). We shall use the notation from ([3]). Gluing together the ends of the gaps in J, we obtain the lexicographically ordered square. This, together with the constructions of ([3]), gives a representation of the space Y_i $(i = 1, 2)$ as the image of a zero-dimensional bicompactum with the first axiom of countability: $\Phi : \theta = J \times K \to Y_i$, $i = 1, 2$. After this, in ([3]), from the disjoint sum of the one-dimensional in all senses bicompacta Y_1 and Y_2, after gluing by Φ, a two-dimensional in the inductive senses bicompactum X is obtained.

Let s be a point not belonging to θ. In the product $C = C_0 \times (\{s\} \cup \theta)$ introduce a topology in the following way. The set $\{c\} \times \theta$, $c \in C_0$, will be regarded as an open-and-closed subspace in the space C with topology θ. A neighborhood of the point (c_0, s), $c_0 \in C_0 \subset C$, will be called a set $l \times (\{s\} \cup \theta)$, where l is a neighborhood of the point c_0 in C_0, and also any set obtained from this by throwing out a finite number of sets of the form $\{c\} \times \theta$, $c \in C_0$.

It is not difficult to construct in the square I^2 a countable base D consisting of rectangles (of the form $l_1 \times l_2$, $l_1, l_2 \subset I^2$) of diameter less than $1/4$, such that the boundaries of any two intersect in no more than four points that are not vertices of squares, and no point of the square I^2 belongs to more than two boundaries. Let Q be the set of points of the square I^2 belonging to more than one boundary. It is, as is easy to see, countable.

Take a point $q \in Q$. It lies at the intersection of the boundaries of two squares d_0 and d_1 of the family D. Let l_i, $i = 0, 1$, be the ray emanating from the point q, passing along the boundary of the square d_i. Let t_i, $i = 0, 1$, be the part lying in the square I^2 of the closed sector with vertex at q, containing the ray l_i and bounded by rays forming angles $\pm 30^\circ$ with l_i. The intersection of these sectors, as is easily seen, consists only of their common vertex q. The triple (q, t_0, t_1) will be called marked. Denote the set of marked triples by Σ_1.

A pair (x, N), $x \in I^2$, $N \subseteq I^2$, will be called marked if: a) $N = I^2$, if the point x does not belong to the boundary of any square of the base D; b) $N = [d]$

* The bicompactum T^3 from ([4]).

or $I^2 \setminus d$, if the point $x \notin Q$ lies on the boundary of a square $d \in D$; c) $N = [d_1] \cap [d_2]$, or $[d_1] \cap (I^2 \setminus d_2)$, or $(I^2 \setminus d_1) \cap [d_2]$, or $(I^2 \setminus d_1) \cap (I^2 \setminus d_2)$, where d_1, d_2 are distinct elements of the base D, in the intersection of whose boundaries the point x lies, if $x \in Q$. We denote the set of marked pairs by Σ_2. As is easy to see, the set $\Sigma = \Sigma_1 \cup \Sigma_2$ has the same cardinality as the interval I. Let $f : I \to \Sigma$ be a one-to-one correspondence.

In the product $I^2 \times C$, endowed with the Tychonoff topology, we perform the following rearrangements: a) if $f(r_2)$ is a marked pair (x, N), then from the space we remove the sets $(I^2 \setminus N) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, and in the sets $\{x\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, we perform the decompositions ψ_i, and then glue these spaces $\psi_i(\theta)$, $i = 0, 1$, by Φ and obtain a space homeomorphic to X; b) if $f(r_2)$ is a marked triple (q, t_0, t_1), then from the space we remove the sets $(I^2 \setminus t_i) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, and in the sets $\{q\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, we perform the decompositions ψ_i, and then these spaces $\psi_i(\theta)$, $i = 0, 1$, are glued by Φ, and we obtain a space homeomorphic to X. What is obtained under these rearrangements from a set $M \subset I^2 \times C$ will be denoted by $\Psi(M)$. Let $\Psi(I^2 \times C) = R$.

The dimension estimates for the space R are obtained as follows. Since R contains sets homeomorphic to a square, all dimensions of R are not less than 2. By tracing the images under Ψ of cubic neighborhoods in $I^2 \times C$ and their finite unions, it is not difficult to verify that all dimensions are not greater than 3.

We show that $\operatorname{ind} R \le 2$. For points of $\Psi((r_1, r_2, i) \times \theta)$, $i = 0, 1$, this estimate is obvious. As is easy to see, the set $\operatorname{Int} \Psi(l \times [d])$, where $d \in D$, l is a clopen set in C, has a one-dimensional boundary. Altogether this gives the required estimate.

We show that $\operatorname{Ind} R \ge 3$. By standard arguments (see (3)) one can show that any partition between the sets $\Psi((\{0\} \times I) \times C)$ and $\Psi((\{1\} \times I) \times C)$ contains either a set open in some subspace of the form $\Psi(I^2 \times \{(r_1, r_2, i)\})$, $i = 0$ or 1, or some set of the form $[\Psi(E \times (\{r_1\} \times I \times D \times (\{1\} \cup \theta))) \cap R^0]$, where R^0 is the set of points of R at which the open mapping $\pi' : R \to I^2$, generated by the projection onto the factor $\pi : I^2 \times C \to I^2$, is open, and E is a partition between $\{0\} \times I$ and $\{1\} \times I$ in I^2. As is easy to see, in both cases the dimension of the boundary is not less than 2, and therefore $\operatorname{Ind} R \geq 3$.

§ 2. **A bicompactum R_2 with $\dim R_2 = 1$, $\operatorname{ind} R_2 = 2$, $\operatorname{Ind} R_2 = 3$.** In the Sierpiński carpet S, obtained from the square I^2 by removing a countable family Δ of open squares (see (1)), one can choose a countable base D, consisting of intersections with S of rectangles lying in I^2 of diameter less than $1/4$, such that: a) the boundaries of any two rectangles intersect in at most four points that are not their vertices; b) no point of the carpet S belongs to the boundaries of more than two elements of the base D; c) the boundaries of the elements of the base D are connected; d) the boundary of an element of the base D intersects the boundary of a square of the family Δ in two points or does not intersect it at all.

Let Q be the set of points of the carpet S belonging to the boundaries of more than one element of the family D. It is, as is easy to see, countable.

To each element δ of the family Δ we assign a copy $X(\delta)$ of the space X from (3). On the set

$$S^* = S \cup \left(\bigcup_{\delta \in \Delta} X(\delta) \right)$$

we introduce a topology as follows. The subspaces $X(\delta)$ will be considered clopen. A neighborhood of a point from $S \subset S^*$ will be a set obtained from an element d of the family D by adjoining those sets $X(\delta)$ for which the boundary of the square δ lies entirely in d.

Let the symbols $\psi_i, \theta, Y_i, \Phi, X, C$ denote the same as in § 1. By marked pairs (x, N) and marked triples (q, t_0, t_1) we shall mean pairs and triples described in the same way as in § 1, with the difference that, instead of N, t_0, t_1 from § 1, their intersections with S are taken, with the addition of those sets ···

sets $X(\delta)$ for which the boundary of the square δ lies entirely in this intersection. Let $f : I \to \Sigma$ be a one-to-one correspondence between the points of the segment and the set Σ of all marked pairs and triples.

In the product $S^* \times C$, taken in the Tikhonov topology, we make the following rearrangements: a) if $f(r_2)$ is a marked pair (x, N), then we throw out from the space the sets $(S^* \setminus N) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, in the sets $\{x\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, perform the decompositions ψ_i, and then glue these spaces $\psi_i(\theta)$, $i = 0, 1$, by Φ and obtain a space homeomorphic to X; b) if $f(r_2)$ is a marked triple (q, t_0, t_1), then we throw out from the space the sets $(I^2 \setminus t_i) \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, in the sets $\{q\} \times ((r_1, r_2, i) \times \theta)$, $i = 0, 1$, perform the decompositions ψ_i, and then glue these spaces $\psi_i(\theta)$, $i = 0, 1$, by Φ and obtain a space homeomorphic to X.

The estimates of the dimensions of the bicompactum R_3 obtained after these rearrangements are made in the same way as in § 1: $\dim R_2 = 1$, $\operatorname{ind} R_2 = 2$, $\operatorname{Ind} R_2 = 3$.

§ 3. Construction, from a bicompactum T with $\dim T = l$, $\operatorname{ind} T = m$, $\operatorname{Ind} T = n$, $1 \le l \le m \le n$, of a bicompactum T^* with $\dim T^* = l$, $\operatorname{ind} T^* = m + 1$, $\operatorname{Ind} T^* = n + 1$, where $T^* = T_1 \cup T_2$, with T_i, $i = 1, 2$, bicompacta satisfying $\dim T_i = l$, $\operatorname{ind} T_i = m$, $\operatorname{Ind} T_i = n$.

In the square I^2 one can find a vertical $\{\vartheta\} \times I$ which does not intersect the closures of the elements of the family Δ of squares thrown out of the square I^2 in the construction of the Sierpiński carpet S. To each element δ of the family Δ we assign a copy $T(\delta)$ of the bicompactum T. On the set

$$S^* = S \cup \left(\bigcup_{\delta \in \Delta} T(\delta) \right)$$

we introduce a topology in the same way as in the analogous situation in § 2.

Let $\varphi_i : \theta \to T$ be a mapping of some zero-dimensional bicompactum θ onto the bicompactum T such that for every point $t \in T$ (respectively, every closed set $T' \subseteq T$) there is an arbitrarily small neighborhood V with $(m-1)$-dimensional (respectively, $(n-1)$-dimensional) boundary such that the set $\varphi^{-1}([V])$ is open. At least one such bicompactum θ exists, for example, the absolute of the bicompactum T.

Let, as before, $C_0 = I \times I \times D$ carry the lexicographic order. In the set $C = C_0 \cup (I \times I \times \theta)$ we introduce a topology in the following way. The set $\{r_1\} \times \{r_2\} \times \theta$ will be regarded as an open-and-closed subspace with topology θ. Convergence to points from C_0 is interval convergence, taking into account that the set $\{r_1\} \times \{r_2\} \times \theta$ lies between the points $(r_1, r_2, 0)$ and $(r_1, r_2, 1)$.

We shall call marked a pair (x, N), where $x = (\vartheta, r) \in S$, and N is obtained from one of the sets

$$N_0 = ([0, \vartheta] \times [0, r] \cup [\vartheta, 1] \times [r, 1]) \cap S$$

or

$$N_1 = ([0, \vartheta] \times [r, 1] \cup [\vartheta, 1] \times [0, r]) \cap S$$

by adding those sets $T(\delta)$ for which the boundary of the square δ intersects the corresponding set N_i.

Let $f : I \to \Sigma$ be a one-to-one correspondence between the points of the segment and the set Σ of all marked pairs.

In the Tikhonov product $S^* \times C$ we make the following rearrangements: if $f(r_2) = (x, N)$, then we throw out from the space the set $(S^* \setminus N) \times (\{r_1\} \times \{r_2\} \times \theta)$, and in the set $\{x\} \times (\{r_1\} \times \{r_2\} \times \theta)$ we perform the identification φ. The estimates of the dimensions of the bicompactum T^* obtained as a result of these rearrangements Φ and of its closed subspaces $T_i = \Phi(S_i^* \times C)$, $i = 1, 2$, where S_1^* and S_2^* are the sets of points of the bicompactum S^* lying respectively not to the right and not to the left of the vertical $\{\vartheta\} \times \Gamma$, are carried out as in § 1.

§ 4. **Construction, from a bicompactum T^* with $\dim T^* = l$, $\operatorname{ind} T^* = m + 1$, $\operatorname{Ind} T^* = n + 1$, representable as the union of two of its closed subsets $T^* = T_1 \cup T_2$ with $\dim T_i = l$, $\operatorname{ind} T_i = m$, $\operatorname{Ind} T_i = n$, $i = 1, 2$, of a bicompactum T^{**} with $\dim T^{**} = l$, $\operatorname{ind} T^{**} = m + 1$, $\operatorname{Ind} T^{**} = n + 2$.**

Let $\varphi : \theta \to T$ be a continuous mapping of some zero-dimensional bicompactum θ onto a bicompactum T such that at each point $t \in T$ (respectively, at each closed set $T' \subseteq T$) there is an arbitrarily small neighborhood with m-dimensional (respectively n-dimensional) boundary such that the set $f^{-1}([V])$ is open. We shall also assume that $\theta_1 \cap \theta_2 = \emptyset$, where $\theta_i = \varphi^{-1}(T_i)$, $i = 1, 2$.

Let C denote (taking into account the change in the meaning of θ) the same thing as in § 3. Let D be the base in the Sierpiński carpet S described in § 2. Let the bicompactum S^* be obtained from the carpet S in the same way as in § 2, with the only difference that, instead of the set $X(\delta)$, we shall add copies $T^*(\delta)$ of the bicompactum T^*. The set Q, the marked pairs, triples, and the correspondence $f : I \to \Sigma$ are the same as in § 2, taking into account the difference in the construction.

In the product $S^* \times C$, taken in the Tikhonov topology, we make the following rearrangements: a) if $f(r_2)$ is a marked pair (x, N), then we remove from the space the sets $(S^* \setminus N) \times (\{r_1\} \times \{r_2\} \times \theta)$ and in the set $\{x\} \times (\{r_1\} \times \{r_2\} \times 0)$ make the identification φ; b) if $f(r_2)$ is a marked triple (q, t_0, t_1), then we remove from the space the sets $(I^2 \setminus t_i) \times (\{r_1\} \times \{r_2\} \times \theta_i)$, $i = 1, 2$, and in the set $\{x\} \times (\{r_1\} \times \{r_2\} \times \theta)$ carry out the identification φ.

The estimates of the dimensions of the bicompactum T^{**} obtained as a result of these rearrangements are made in the same way as in the preceding cases.

§ 5. **A bicompactum R_i with $\dim R_i = 1$, $\operatorname{ind} R_i = i$, $\operatorname{Ind} R_i = 2i - 1$.** As the bicompactum R_1 we shall take the interval I. Having first carried out with it (as T) the constructions of § 3, and then the constructions of § 4, we obtain a bicompactum T^{**} with $\dim T^{**} = 1$, $\operatorname{ind} T^{**} = 2$, $\operatorname{Ind} T^{**} = 3$. We shall take this bicompactum as R_2. Continuing this process, we obtain bicompacta R_i for all i. As is easy to see (this is proved by induction), starting with the interval, we can arrange that in all the bicompacta the first axiom of countability is satisfied.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.