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Abstract

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TOPOLOGICAL STRUCTURE OF NORMALLY SOLVABLE ELLIPTIC OPERATORS WITH CONSTANT COEFFICIENTS IN THE PLANE

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The differential operator

$$I = \sum_{j=0}^{n} A_j \frac{\partial^n}{\partial x^{n-j} \partial y^j} + \sum_{k+l<n} B_{kl} \frac{\partial^{k+l}}{\partial x^k \partial y^l}, \tag{1}$$

where A_j, B_{kl} are constant real $m \times m$ matrices, is called elliptic if the corresponding matrix pencil $L(\lambda)$ of the form

$$L(\lambda) = \lambda^n A_0 + \lambda^{n-1} A_1 + \cdots + A_n \tag{2}$$

satisfies one of the conditions:

$$\det A_0 > 0, \qquad \det L(\lambda) > 0, \qquad -\infty < \lambda < +\infty, \tag{3}$$

or

$$\det A_0 < 0, \qquad \det L(\lambda) < 0, \qquad -\infty < \lambda < +\infty. \tag{3'}$$

The sets of operators satisfying conditions (3) and (3′) will be denoted by $\mathcal{L}_+(n,m)$ and $\mathcal{L}_-(n,m)$, respectively. As we showed in [1], each of the sets (3) and (3′), for $m > 2$, consists of two connected components. The components of $\mathcal{L}_+(n,m)$ will be denoted by $l_1^+(n,m)$ and $l_2^+(n,m)$. As we showed in [1], every operator from $l_1^+(n,m)$ is homotopic to the operator

$$I_{01} = \Delta^{n/2} E_m \qquad \text{for even } n,$$

$$I_{11} = \Delta^q \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \qquad \text{for odd } n, \quad n = 2q+1,$$

where

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2; \qquad K = \begin{pmatrix} \partial/\partial x & \partial/\partial y \\ -\partial/\partial y & \partial/\partial x \end{pmatrix}.$$

We shall now indicate a necessary and sufficient condition under which the operator (1) from $\mathcal{L}_+(n,m)$ belongs to the first component $l_1^+(n,m)$. To this end, with the operator (1) we associate an $n \cdot m \times n \cdot m$ matrix $\mathfrak{A}$:

$$\mathfrak{A} = \begin{pmatrix} 0 & E & 0 & \cdots & 0 \\ 0 & 0 & E & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & E \\ -A_0^{-1}A_n & \cdots & \cdots & \cdots & -A_0^{-1}A_1 \end{pmatrix}. \tag{4}$$

The spectrum of the pencil (2) coincides with the spectrum of the matrix $\mathfrak{A}$. Let the $n \cdot m/2$-vectors

$$f_1, f_2, \dots, f_r, \qquad r = n \cdot m/2, \tag{5}$$

form a basis in the invariant subspace of the matrix $\mathfrak{A}$ corresponding to the spectrum from the upper half-plane. Consider the determinant

$$\delta_L = (i)^r \det(f_1, f_2, \dots, f_r, \bar{f}_1, \bar{f}_2, \dots, \bar{f}_r), \tag{6}$$

where $i = \sqrt{-1}$. It is not hard to prove that δ_L is real.

Theorem 1. *In order that an elliptic operator of the form (1) from $\mathcal{L}_+(n,m)$ belong to the component $l_1^+(n,m)$, it is necessary and sufficient that, for even n $(n = 2q)$,*

$$\operatorname{sign} \delta_L = (-1)^{q(q-1)m/2} \tag{7}$$

and, for odd n $(n = 2q+1)$,

$$\operatorname{sign} \delta_L = (-1)^{m(q^2-q-m)/2}. \tag{7'}$$

Fig. 1

Figure 1: Fig. 1

We outline the proof of the theorem. Let $L_\varepsilon(\lambda)$ $(0 \le \varepsilon \le 1)$ be a deformation of the bundle $L_0(\lambda)$ corresponding to an operator from $\mathcal{L}_+(n,m)$. The basis (5) corresponding to $L_\varepsilon(\lambda)$ can be constructed so as to depend continuously on $\varepsilon \in [0,1]$; consequently, $\delta_{L_0} = \delta_{L_1}$. It is not hard to calculate that the δ_L corresponding to I_{01} and I_{11} are computed by the formulas (7) and (7′), and, moreover, $\delta_{L_{01}} = -\delta_{L_{02}}$, $\delta_{L_{11}} = -\delta_{L_{22}}$, where $L_{02}^{(\lambda)}, L_{22}^{(\lambda)}$ are the bundles corresponding to the canonical representatives from $l_2^+(n,m)$ for odd n (see (1)).

Fig. 1

2. Let G be a domain in the (x,y)-plane, bounded by an infinitely smooth Jordan curve Γ. Denote by $\mathcal{H}_l(G)$, where l is an integer $> n/2$, the direct product of m spaces

$$\mathcal{H}_{l-1/2}(\Gamma) \times \mathcal{H}_{l-1-1/2}(\Gamma) \times \dots \times \mathcal{H}_{l-n/2-1/2}(\Gamma).$$

For the operator (1) of even order $n = 2q$, consider the Dirichlet problem:

$$I(u) = f, \qquad f \in \mathcal{H}_{l-n}(G), \tag{8}$$

$$u|_\Gamma = g_0, \qquad \partial u/\partial\nu|_\Gamma = g_1, \dots, \partial^{q-1}u/\partial\nu^{q-1}|_\Gamma = g_{q-1}, \tag{9}$$

where $\partial/\partial\nu$ is differentiation along the normal to the boundary Γ, and $g_j \in \mathcal{H}_{l-j-1/2}$.

The operator from $\mathcal{H}_l(G)$ to $\mathcal{H}_{l-g-1/2}(\Gamma)$, defined by the equalities (9), will be denoted by D.

We shall call problem (7)–(8) normally solvable if the operator $N = (I, D)$, acting from $\mathcal{H}_l(G)$ to $\mathcal{H}_{l-n}(G) \times \mathcal{H}_{l-g-1/2}(\Gamma)$, is Noetherian, i.e., the kernel and cokernel of the operator N have finite dimensions and the range of N is closed in $\mathcal{H}_{l-n}(G) \times \mathcal{H}_{l-g-1/2}(\Gamma)$ (see ([2])).

Without loss of generality, we shall assume that all eigenvalues of the bundle (2) are simple. Then, as can be proved using the result of Lopatinskii (see ([3])), in order that problem (8)–(9) be normally solvable, it is necessary and sufficient that

$$d = \det \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_r \\ \lambda_1\psi_1 & \lambda_2\psi_2 & \cdots & \lambda_r\psi_r \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{q-1}\psi_1 & \lambda_2^{q-1}\psi_2 & \cdots & \lambda_r^{q-1}\psi_r \end{pmatrix} \neq 0, \tag{10}$$

where $\lambda_1, \lambda_2, \dots, \lambda_r$ are the eigenvalues of the bundle (2) in the upper half-plane, and $\psi_1, \psi_2, \dots, \psi_r$ are the corresponding eigenvectors.

Theorem 2. *The set of elliptic operators (1) belonging to $\mathcal{L}_+(n,m)$, for which the Dirichlet problem is normally solvable, splits into two connected components: $\tilde{l}_1^+(n,m)$ and $\tilde{l}_2^+(n,m)$; moreover, $\tilde{l}_1^+(n,m)$ and $\tilde{l}_2^+(n,m)$ are open everywhere dense connected subsets of the components $l_1^+(n,m)$ and $l_2^+(n,m)$, respectively.*

We outline the proof of this theorem. Let, for definiteness, the matrix bundles $L_1(\lambda)$ and $L_2(\lambda)$ correspond to operators from the component $l_2^+(n,m)$, and let condition (10) be fulfilled for both of these bundles.

Using the representation of $L_1(\lambda)$ and $L_2(\lambda)$ as products of linear bundles (see (1)), one can prove that there exists a matrix bundle $L_\varepsilon(\lambda)$, analytically dependent on ε, $0 \leqslant \varepsilon \leqslant 1$, belonging to $l_1^+(n,m)$ for all ε:

$$L_\varepsilon(\lambda)\big|_{\varepsilon=0} = L_1(\lambda); \qquad L_\varepsilon(\lambda)\big|_{\varepsilon=1} = L_2(\lambda)$$

and such that the eigenvalues $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \dots, \lambda_r(\varepsilon)$, $\operatorname{Im}\lambda_j(\varepsilon) > 0$, and the corresponding eigenvectors $\psi_1(\varepsilon), \psi_2(\varepsilon), \dots, \psi_r(\varepsilon)$ are analytic functions in a neighborhood of the segment $[0,1]$ of the complex ε-plane. The function $d(\varepsilon)$ (see (10)) is analytic, is not identically equal to zero, and therefore has at most a finite number of zeros on the segment $[0,1]$. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ be the zeros of $d(\varepsilon)$ on the segment $[0,1]$; we can bypass them in the complex ε-plane by varying ε along a contour γ connecting the points 0 and 1 (see Fig. 1), where the semicircles with centers at the zeros $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ have sufficiently small radius. We construct a bundle depending on ε, with eigenvalues $\lambda_1(\varepsilon), \overline{\lambda}_1(\varepsilon), \dots, \lambda_r(\varepsilon), \overline{\lambda}_r(\varepsilon)$ and the corresponding eigenvectors $\psi_1(\varepsilon), \overline{\psi}_1(\varepsilon), \dots, \psi_r(\varepsilon), \overline{\psi}_r(\varepsilon)$, where ε varies from 0 to 1 along the contour γ. This bundle, as is not difficult to show, is real for all $\varepsilon \in [0,1]$ and solves the indicated problem.

We note that each of the connected components $l_1^+(n,m)$ and $l_2^+(n,m)$ also contains operators for which the Dirichlet problem is not normally solvable. For example, the operator

$$\begin{pmatrix} B_0 & 0 \\ 0 & B_0 \end{pmatrix},$$

where B_0 is the Bitsadze operator (see (*))

$$B_0 = \begin{pmatrix} \partial^2/\partial x^2 - \partial^2/\partial y^2 & 2\partial^2/\partial x\,\partial y \\ -2\partial^2/\partial x\,\partial y & \partial^2/\partial x^2 - \partial^2/\partial y^2 \end{pmatrix},$$

belongs to $l_1^+(n,m)$ and, consequently, is homotopic to the operator ΔE_4. This assertion contains the answer to a question posed by I. M. Gelfand (see (5)).

By the same method one can prove that if the Dirichlet conditions on the boundary Γ are replaced by arbitrary conditions with coefficients independent of the point of the boundary, then the set of operators (1) from $\mathcal{L}^+(n, m)$ for which this boundary-value problem is normally solvable splits into two connected components.

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Note: Figure translations are in progress. See original paper for figures.